Coordinating Charitable Donations

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Charity is typically carried out by individual donors, who donate money to charities they support, or by centralized organizations such as governments or municipalities, which collect individual contributions and distribute them among a set of charities. Individual charity respects the will of the donors, but may be inefficient due to a lack of coordination; centralized charity is potentially more efficient, but may ignore the will of individual donors. We present a mechanism that combines the advantages of both methods for donors with Leontief preferences (i.e., each donor seeks to maximize an individually weighted minimum of all contributions across the charities). The mechanism distributes the contribution of each donor efficiently such that no subset of donors has an incentive to redistribute their donations. Moreover, it is group-strategyproof, satisfies desirable monotonicity properties, maximizes Nash welfare, returns a unique Lindahl equilibrium, can be computed efficiently, and implemented via natural best-response spending dynamics.

Keywords: Mechanism design, collective decision making, Leontief preferences, public good markets, spending dynamics

1. Introduction

Private charity, given by individual donors to underprivileged people in their vicinity, has existed long before institutionalized charity via municipal or governmental organizations. Its main advantage is transparency—the donors know exactly where their money goes to, which may increase their willingness to donate. A major disadvantage of private charity is the lack of coordination: donors may donate to certain people or charities without knowing that these recipients have already received ample money from other donors. Centralized charity via governments or municipalities is potentially more efficient but, if not done carefully, may disrespect the will of the donors.

Example 1. Suppose there are two donors and four charities. The first donor is willing to contribute \$900 and supports charities A, B, and C, whereas the second donor is willing to contribute \$100 and supports charities C and D.

- A central organization may collect the contributions of the donors and divide them equally among the four charities, so that each charity receives \$250. While this outcome is the most balanced possible for the charities, it goes against the will of the first donor, since \$150 of her contribution is used to support charity D.

- By contrast, without any coordination, each donor may split her individual contribution equally between the charities that she approves. As a result, charities A and B receive \$300 each, charity C receives \$350, and charity D receives \$50. However, if the second donor knew that charity C would already receive \$300 from the first donor, she would probably prefer to donate more to charity D, for which she is the only contributor.

- Our suggested mechanism would give \$300 to each of charities A, B, and C, and \$100 to charity D. This distribution can be understood as recommendations to the individual donors: the first donor should distribute her contribution uniformly over charities A, B, and C whereas the second donor should transfer all her contribution to charity D. Importantly, the contribution of each donor only goes to charities that the donor approves. Subject to that, the donations are divided as equally as possible.

Evaluating and comparing donor coordination mechanisms requires some assumptions on the donors' preferences. Since charitable giving is often driven by egalitarian considerations, we assume that donors want to *maximize the minimum amount* given to a charity they approve. This can be formalized by endowing each donor with a utility function mapping each distribution to the smallest amount of money allocated to one of the donor's approved charities. For example, for the distribution $(300, 300, 300, 100)$, the first agent's utility is 300 and the second agent's utility is 100. More generally, our model allows donors to attribute different values than merely 1 and 0 (which indicate approval and disapproval, respectively) to different charities. If a donor i values a charity x at $v_{i,x}$, then i's utility from a distribution δ equals $\min_x \delta(x)/v_{i,x}$, where the minimum is taken over all charities x for which $v_{i,x} > 0$. Such utilities are known as *Leontief utilities* (see, e.g., Varian, 1992; Mas-Colell et al., 1995) and are often studied in resource allocation problems (e.g., Codenotti and Varadarajan, 2004; Nicoló, 2004; Ghodsi et al., 2011; Li and Xue, 2013). Whenever $v_{i,x} \in \{0, 1\}$ for all agents i and charities x, we refer to this as (Leontief) utility functions with *binary weights*.[1]

Given the contribution and utility function of each donor, our goal is to distribute the money among the charities in a way that respects the individual donors' preferences. The idea of "respecting the donors' preferences" is captured by the notion of an *equilibrium distribution*. We say that a distribution is *in equilibrium* if it can be implemented by telling each donor how to distribute her contribution among the charities, such that the prescribed distribution maximizes the donor's utility given that the other donors follow their own prescriptions. One can check that, in Example 1, the unique equilibrium distribution is $(300, 300, 300, 100)$.

Since utility functions are concave, it follows from a result by Rosen (1965) that an equilibrium distribution (in pure strategies) always exists. Our first main result is that each preference profile admits a *unique* equilibrium distribution. Moreover, we prove that

[1] One can further assume that, subject to maximizing the minimum amount given to an approved charity, the donors want to maximize the second-smallest amount, then the third-smallest amount, and so on. Our results carry over to this class of preferences; see Section 8.

the unique equilibrium distribution coincides with the unique distribution that maximizes the product of individual utilities weighted by their contributions (*Nash welfare*), which implies that it is Pareto efficient, and can be computed via convex programming. The equilibrium distribution can be viewed as the market equilibrium of a pure public good market as well as a Lindahl equilibrium with personalized prices.

In Example 1, the equilibrium distribution $(300, 300, 300, 100)$ also maximizes the minimum utility of all agents (*egalitarian welfare*) subject to each donor only contributing to her approved charities. We show that this is true in general when weights are binary, and extends to an infinite class of welfare measures "in between" Nash welfare and egalitarian welfare. Moreover, for the case of binary weights, we show that the equilibrium distribution coincides with the distribution that allocates individual contributions to approved charities such that the minimum contribution to charities is maximized lexicographically. This allows for simpler computation via linear programming.

Based on existence and uniqueness, we can define the *equilibrium distribution rule (EDR)*—the mechanism that returns the unique equilibrium distribution of a given profile. Our second set of results show that *EDR* exhibits remarkable axiomatic properties:

- *Group-strategyproofness*: agents and coalitions thereof are never better off by misrepresenting their preferences, and are strictly better off by contributing more money,

- *Preference-monotonicity*: the amount donated to a charity can only increase when agents increase their valuation for the charity, and

- *Contribution-monotonicity*: the amount donated to a charity can only increase when agents increase their contributions.

As we further show, equilibrium distributions are the limit distributions of natural spending dynamics based on best responses. This can be leveraged in settings where a central infrastructure is unavailable or donors are reluctant to completely reveal their preferences. One could envision a scenario in which donors have set aside a, say, monthly budget to spend on charitable activities and repeatedly distribute this budget after observing the donations made by other donors in previous rounds. We prove that, when donors spend their money myopically optimally in each round, the relative overall distribution of donations converges to the equilibrium distribution. Hence, socially desirable outcomes can be attained even without a central infrastructure, as long as charities are transparent about the donations they receive. This scenario also allows for occasional changes in the agents' preferences and contributions, as the process keeps converging towards an equilibrium distribution of the current profile.

Apart from private charity, our results are also applicable to donation programs—prominent examples include *AmazonSmile* and government programs (e.g., *cinque per mille* run by the Italian Revenue Agency and *mechanizm 1%* in Poland). In these programs, participants can redirect a portion of their payments (purchase price and income tax, respectively) to charitable organizations of their choice.[2] AmazonSmile ran

[2]These programs only allow each participant to choose exactly one charitable organization. However,

from 2013 to 2023 and was used to allocate a total of $400 million. In 2022, a record €510 million were distributed via cinque per mille in Italy, and Poland increased the donatable quota of personal income tax from 1% to 1.5%.[3]

In contrast to private charity, participants of donation programs do not have the option of taking their money out of the system, which means that the important issue lies in finding a desirable distribution of the contributions rather than in incentivizing the participants to donate in the first place. A major criticism of the Polish program is that large organizations accumulate most of the donations whereas some locally popular charities are left almost empty-handed.[4] The same phenomenon is well-known in healthcare, where organizations helping people with rare diseases find it difficult to attract donors. This issue is alleviated by our assumption of Leontief preferences as illustrated by the following example.

Example 2. Suppose there are ten donors, each of whom donates $30. Donor i assigns value 2 to a charity A that supports patients with a common disease, and assigns value 1 to a charity B_i that supports patients with some rare disease.

If each donor is forced to select a single charity to donate to (as in the redirection programs mentioned above), then A will receive all donations and $B_1, \dots, B_{10}$ will receive none. When donors can distribute their donations independently, without coordination, they will likely split their donations in the ratio $2:1$. As a result, A will get $200 and each B_i will get only $10.

By contrast, our suggested mechanism will give $50 to A and $25 to each B_i, which is the unique equilibrium. This distribution adequately reflects the donors' preferences, as each of them supports A twice as much as B_i.

The remainder of this paper is structured as follows. After discussing related work in Section 2, we formally introduce our model in Section 3. Section 4 lays the foundation for the proposed distribution rule by showing existence and uniqueness of equilibrium distributions via Nash welfare maximality as well as characterizing Pareto efficient distributions and pointing out connections to public good markets and Lindahl equilibrium. Subsequently, we define *EDR* as the rule that always returns the equilibrium distribution and examine it axiomatically in Section 5. In Section 6, we explore natural spending dynamics that converge towards the equilibrium distribution. The special case of Leontief utilities with binary weights allows for alternative characterizations of *EDR* that enable its computation via linear programming, as well as further justification of *EDR* via a wide class of welfare functions; this is covered in Section 7. The paper concludes in Section 8

as Brandl et al. (2022) argued, permitting them to indicate support for multiple organizations can increase the efficiency of the distribution.

[3]Leontief utility functions are not only suitable in the context of charity but also in other settings where agents have to jointly fund resources that are complementary in nature. For example, consider a communication network and a set of agents, each of whom intends to transmit a signal along an individual path in the network. Their utilities are given by the quality of the signal at the last node on their path, which equals the minimal transmission quality of an edge along that path. Our mechanism can be used to coordinate agents' investments to improve the transmission quality of edges.

[4]See `https://pl.wikipedia.org/wiki/Przekazywanie_1%25_podatku_dochodowego_na_rzecz_organizacji_po%C5%BCytku_publicznego_w_Polsce` and the references therein.

with a brief discussion of alternative utility models such as linear, Cobb-Douglas, and leximin Leontief utilities. Elaborate proofs are deferred to the Appendix.

2. Related work

A well-studied problem related to the one we study in this paper is that of *private provision of public goods* (see, e.g., Samuelson, 1954; Bergstrom et al., 1986; Varian, 1994; Falkinger, 1996; Falkinger et al., 2000). In this stream of research, each agent decides on how much money she wants to contribute to funding a public good. Typically, this leads to under-provision of the public good in equilibrium, resulting in inefficient outcomes. In our model, we assume that agents have already set aside a budget to support public charities, either voluntarily or compulsorily (as part of their taxes or payments to a company). The inefficiency that we are worried about is an inefficient allocation among different public goods. As a result, the problem we study has the flavor of both social choice and fair division.

Socially optimal outcomes can be implemented by well-known strategyproof mechanisms such as the Vickrey-Clarke-Groves (VCG) mechanism. However, VCG fails to be budget-balanced: it collects money from the agents, and has to 'burn' that money in order to maintain strategyproofness. By contrast, in our setting, the monetary contribution of each agent is fixed and independent of the agent's preferences. The entire contribution goes to charities approved by the agent and the central issue is one of fair distribution. As shown in Section 5.1, strategyproofness can be achieved without imposing additional payments on the agents.

Perhaps the first paper to consider charitable giving from a mechanism design perspective is due to Conitzer and Sandholm (2004, 2011). They let agents incentivize other agents to donate more by devising "matching offers", where a donation is made conditional on how much and to which charities other agents donate. They introduce an expressive bidding language for such offers and study the computational complexity of the resulting market clearing problem.

A rapidly growing stream of research explores *participatory budgeting* (e.g., Aziz and Shah, 2021), which allows citizens to jointly decide how the budget of a municipality should be spent in order to realize projects of public interest. In contrast to charities, the projects considered in participatory budgeting come with a fixed cost (e.g., constructing a new bridge), and each project can be either fully funded or not at all. Moreover, most participatory budgeting papers assume that money is owned by the municipality rather than by the agents themselves.

The work most closely related to ours is that of Brandl et al. (2021, 2022) who initiated the axiomatic study of donor coordination mechanisms. In their model, the utility of each donor is defined as the weighted *sum* of contributions to charities, where the weights correspond to the donor's inherent utilities for a unit of contribution to each charity. Under this assumption, the only efficient distribution in Example 1 is to allocate the entire donation of \$1000 to charity C, since this distribution gives the highest possible utility, 1000, to all donors. However, this distribution leaves charities A, B, and D with no money at all, which may not be what the donors intended. With sum-based utilities, as studied by Brandl et al., charities are perfect *substitutes*: when a donor assigns the same utility to several charities, she is completely indifferent to how money is distributed among these charities. By contrast, in our model of *minimum-based* utilities, charities are perfect *complements*: donors want their money to be evenly distributed among charities they like equally much. Fine-grained preferences over charities can be expressed by setting weights for Leontief utility functions. It can be argued that this assumption better reflects the spirit of charity by not leaving anyone behind. The modified definition of utility functions critically affects the nature of elementary concepts such as efficiency or strategyproofness and fundamentally changes the landscape of attractive mechanisms.

The main result by Brandl et al. (2022) shows that, in their model of linear utilities, the Nash product rule incentivizes agents to contribute their entire budget, even when attractive outside options are available. However, the Nash product rule fails to be strategyproof (Aziz et al., 2020) and violates simple monotonicity conditions (Brandl et al., 2021). In fact, a sweeping impossibility by Brandl et al. (2021) shows that, even in the simple case of binary valuations, no distribution rule that spends money on at least one approved charity of each agent can simultaneously satisfy efficiency and strategyproofness. This confirms a conjecture by Bogomolnaia et al. (2005) and demonstrates the severe limitations of donor coordination with linear utilities. Interestingly, as we show in this paper, Leontief utilities allow for much more positive results.

Originating from the *Nash bargaining solution* (Nash, 1950), the Nash product rule can be interpreted as a tradeoff between maximizing utilitarian and egalitarian welfare, a recurring idea when it comes to finding efficient *and* fair solutions. When allocating divisible *private* goods to agents with linear utility functions, the Nash product rule returns the set of all *competitive equilibria from equal incomes* (Eisenberg and Gale, 1959); thus, it results in an efficient and *envy-free* allocation (Foley, 1967). The connection between market equilibria and Nash welfare maximizers has been extended to various single-seller markets (Jain and Vazirani, 2010). The most prominent of these are *Fisher markets*, in which a set of divisible goods is available in limited quantities, and each buyer has a fixed budget at her disposal. The problem is to find equilibrium prices that clear the market while maximizing each buyer's utility. Eisenberg (1961) shows that when buyers have homogeneous utility functions (which include linear, Cobb-Douglas, and Leontief utility functions), market equilibria for Fisher markets can be found by maximizing the Nash product of individual utility functions (see also Codenotti and Varadarajan, 2007). While this resembles Theorem 1, there are some important differences between Fisher markets and our public good markets. First of all, in contrast to Fisher markets, the relationship between equilibria and Nash welfare maximizers in public good markets does not extend to all homogeneous utility functions. In fact, for linear utility functions, all public good equilibrium distributions may be inefficient (Brandl et al., 2022, Proposition 1). Even for the special case of Leontief preferences, the relationship between equilibria and Nash welfare maximizers seems to be of a different nature. In contrast to our setting (see Section 4.3), Fisher market equilibria with Leontief preferences may involve irrational numbers and thus cannot be computed exactly (Codenotti and

Varadarajan, 2004).[5] Moreover, mechanisms that maximize Nash welfare in private good settings do not share the desirable properties of *EDR*. For example, Ghodsi et al. (2011) show that the Nash product rule violates strategyproofness in a simple resource allocation setting with Leontief preferences. To the best of our knowledge, there is no previous work on Leontief preferences in the context of *public* goods.

A natural special case of our model is that of Leontief utilities with *binary weights*, where agents only approve or disapprove charities and the utility of each agent is given by the minimal amount transferred to any of her approved charities. Under the assumption that agents only contribute to charities they approve and that all individual contributions are equal, this can be interpreted as a (many-to-many) matching problem on a bipartite graph where agents (and their contributions) need to be assigned to charities with unlimited capacity. Bogomolnaia and Moulin (2004) proposed a solution to such matching problems that maximizes egalitarian welfare of the charities (rather than the agents). The intriguing connection between these two types of egalitarianism are addressed in Section 7. Bogomolnaia and Moulin also showed that their solution constitutes a competitive equilibrium from equal incomes (from the charity managers' point of view).

3. The model

Let N be a set of n *agents*. Each agent i contributes an amount $C_i \geq 0$. For every subset of agents $N' \subseteq N$, we denote $C_{N'} := \sum_{i \in N'} C_i$. The sum of all contributions, C_N, is called the *endowment*.

Further, consider a set A of m potential recipients of the contributions, which we refer to as *charities*. A *distribution* is a function δ assigning a nonnegative real number to each charity, such that $\sum_{x \in A} \delta(x) = C_N$. The support $\{x \colon \delta(x) > 0\}$ of δ is denoted by $\mathrm{supp}(\delta)$, and the set of all possible distributions is denoted by $\Delta(C_N)$. For a subset of charities $A' \subseteq A$, we define $\delta(A') := \sum_{x \in A'} \delta(x)$ as the total amount allocated to charities in A'.

For every $i \in N$ and $x \in A$, there is a real number $v_{i,x} \geq 0$ that represents the value of charity x to agent i. We assume that each agent i has at least one charity x for which $v_{i,x} > 0$. For every agent $i \in N$, we define $A_i := \{x \colon v_{i,x} > 0\}$ as the set of charities to which i attributes a positive value.

The utility that agent i derives from distribution δ is denoted by $u_i(\delta)$ and is given by the Leontief utility function:

$$u_i(\delta) = \min_{x \in A_i} \frac{\delta(x)}{v_{i,x}}.$$

Note that, for every charity $x \in A$ and every agent $i \in N$,

$$\delta(x) \geq v_{i,x} \cdot u_i(\delta).$$

[5]Interestingly, Nash welfare maximizers for Fisher markets with linear utilities are always rational-valued, whereas this is not true for our public good markets (Brandl et al., 2022, Table 2).

If all $v_{i,x}$ are in $\{0,1\}$, we refer to Leontief utilities with *binary* weights. A *profile* P consists of $\{C_i\}_{i\in N}$ and $\{v_{i,x}\}_{i\in N, x\in A}$. Throughout this paper, agents with contribution zero do not have any influence on the outcome and can thus be treated as agents who choose not to participate in the mechanism.

A *distribution rule* f maps every profile to a distribution $\Delta(C_N)$ of the total endowment C_N.

4. Equilibrium distributions

The endowment to be distributed consists of the contributions of individual agents. In order to formalize which distributions are in equilibrium, we therefore need to define how distributions can be decomposed into individual distributions.

Definition 1 (Decomposition)**.** A *decomposition* of a distribution δ is a vector of distributions $(\delta_i)_{i\in N}$ with

$$\sum_{i\in N} \delta_i(x) = \delta(x) \qquad \text{for all } x \in A; \tag{1}$$

$$\sum_{x\in A} \delta_i(x) = C_i \qquad \text{for all } i \in N. \tag{2}$$

Clearly, each distribution admits at least one decomposition. We aim for a decomposition in which no agent can increase her utility by changing δ_i, given C_i and the distributions δ_j for $j \neq i$. In other words, we look for a pure strategy Nash equilibrium of the game in which the strategy space of each agent i is the set of δ_i satisfying (2).

Definition 2 (Equilibrium distribution)**.** A distribution δ is *in equilibrium* if it admits a decomposition $(\delta_i)_{i\in N}$ such that, for every agent i and for every alternative distribution δ'_i satisfying $\sum_{x\in A} \delta'_i(x) = C_i$,

$$u_i(\delta) \geq u_i(\delta - \delta_i + \delta'_i).$$

The present section is devoted to proving the following theorem.

Theorem 1. *Every profile admits a* unique *equilibrium distribution. This distribution is Pareto efficient and can be computed via convex programming.*

As a consequence, we can define the *equilibrium distribution rule* as the distribution rule that selects for each profile its unique equilibrium distribution. In Section 5, we will prove that this rule satisfies desirable strategic and monotonicity properties.

4.1. Critical charities

We start by characterizing equilibrium distributions based on critical charities. Given a distribution δ, we define the set of agent i's *critical charities*

$$T_{\delta,i} := \arg\min_{x\in A_i} \frac{\delta(x)}{v_{i,x}}.$$

Each charity $x \in T_{\delta,i}$ is critical for agent i in the sense that the utility of i would decrease if the amount allocated to x were to decrease. Every agent has at least one critical charity. For every agent i and charity x such that either $v_{i,x} > 0$ or $\delta(x) > 0$, the following equivalences hold:

$$\begin{aligned} x \in T_{\delta,i} \quad &\Leftrightarrow \quad \delta(x) = v_{i,x} \cdot u_i(\delta); \\ x \notin T_{\delta,i} \quad &\Leftrightarrow \quad \delta(x) > v_{i,x} \cdot u_i(\delta). \end{aligned} \tag{3}$$

For every group of agents $N' \subseteq N$, we denote by $T_{\delta,N'}$ the set of charities critical to at least one member of N'.

We prove below that a distribution is in equilibrium if and only if each agent contributes only to her critical charities.

Lemma 1. *A distribution δ is in equilibrium if and only if it has a decomposition $(\delta_i)_{i\in N}$ such that $\delta_i(x) = 0$ for every charity $x \notin T_{\delta,i}$. Equivalently, it has a decomposition satisfying the following equality instead of* (2)*:*

$$\sum_{x\in T_{\delta,i}} \delta_i(x) = C_i \qquad \qquad \textit{for all } i \in N. \tag{4}$$

Proof. $\Rightarrow$: Suppose that, in every decomposition of δ, some agent i contributes to a charity $y \notin T_{\delta,i}$. Fix a decomposition $(\delta_i)_{i\in N}$ of δ. Since $\delta(y) > 0$, by (3), $\delta(y) > v_{i,y} \cdot u_i(\delta)$. Agent i can reduce a small amount from $\delta_i(y)$ and distribute it equally among all charities in $T_{\delta,i}$; this strictly increases the Leontief utility of i. Therefore, δ is not an equilibrium distribution.

$\Leftarrow$: Suppose δ has a decomposition in which each agent i only contributes to charities in $T_{\delta,i}$. In every other strategy of agent i, she must contribute less money to at least one such charity, $y \in T_{\delta,i}$. Since $\delta(y) > 0$, by (3), the original distribution to charity y was $\delta(y) = v_{i,y} \cdot u_i(\delta)$, so the new distribution to y is less than $v_{i,y} \cdot u_i(\delta)$. Therefore, the utility of agent i is smaller than $u_i(\delta)$ and the deviation is not beneficial. □

Corollary 1. *In an equilibrium distribution, every charity that receives a positive amount is critical for at least one agent.*

Lemma 1 implies that an equilibrium distribution satisfies an even stronger equilibrium property.

Corollary 2. *In every equilibrium distribution (and associated decomposition), no* group of agents *can deviate without making at least one of its members worse off.*

This is because *any* deviation decreases the contribution to a critical charity of at least one group member. This equilibrium notion is slightly stronger than *strong equilibrium* by Aumann (1959).

4.2. Efficiency

One of the main objectives of a centralized distribution rule is economic efficiency.

Definition 3 (Efficiency)**.** Given a profile P, a distribution $\delta \in \Delta(C_N)$ is *(Pareto) efficient* if there does not exist another distribution $\delta' \in \Delta(C_N)$ that *(Pareto) dominates* δ, i.e., $u_i(\delta') \geq u_i(\delta)$ for all $i \in N$ and $u_i(\delta') > u_i(\delta)$ for at least one $i \in N$. A distribution rule is efficient if it returns an efficient distribution for every profile P.

Corollary 2 implies that every equilibrium distribution is efficient, since any Pareto improvement yields a beneficial deviation for the group of all agents.

The following lemma characterizes efficient distributions of an arbitrary profile.

Lemma 2. *A distribution δ is efficient if and only if every charity $x \in \operatorname{supp}(\delta)$ is critical for some agent.*

Proof. $\Rightarrow$: Suppose that some charity $x \in \operatorname{supp}(\delta)$ is not critical for any agent. Since $\delta(x) > 0$, by (3), $\delta(x) > v_{i,x} \cdot u_i(\delta)$ for all agents $i \in N$. Denote

$$D := \delta(x) - \max_{i \in N} (v_{i,x} \cdot u_i(\delta))$$

where our assumptions imply that $D > 0$. Construct a new distribution δ' by removing $D/2$ from charity x and distributing it equally among all other charities. We claim that $u_i(\delta') > u_i(\delta)$ for every agent $i \in N$. Indeed, if $v_{i,x} = 0$ then u_i does not decrease by the removal from $\delta(x)$, and strictly increases by the addition to all other charities. Otherwise,

$$u_i(\delta') = \min\left(\frac{\delta'(x)}{v_{i,x}}, \min_{y \in A_i \setminus x} \frac{\delta'(y)}{v_{i,y}}\right).$$

Both terms are larger than $u_i(\delta)$:

- The former term is $(\delta(x) - D/2)/v_{i,x} > (\delta(x) - D)/v_{i,x} = (\max_{j \in N}[v_{j,x} \cdot u_j(\delta)])/v_{i,x} \geq u_i(\delta)$ by construction.
- For the latter term, the fact that $u_i(\delta) < \delta(x)/v_{i,x}$ implies that $u_i(\delta) = \min_{y \in A_i \setminus x} (\delta(y)/v_{i,y})$, and $\min_{y \in A_i \setminus x} (\delta'(y)/v_{i,y})$ is strictly larger than that since each charity $y \in A \setminus x$ receives additional funding in δ'.

Hence, δ is not efficient.

$\Leftarrow$: Suppose that every charity $x \in \operatorname{supp}(\delta)$ is critical for some agent. Let δ' be any distribution different than δ. Since the sum of both distributions is the same (C_N), there exists a charity $y \in \operatorname{supp}(\delta)$ with $\delta'(y) < \delta(y)$. Let $i_y \in N$ be an agent for whom y is critical in δ. Then the utility of i_y is strictly smaller in δ':

$$\begin{aligned}
u_{i_y}(\delta') &\leq \frac{\delta'(y)}{v_{i_y,y}} && \text{(by definition of Leontief utilities)} \\
&< \frac{\delta(y)}{v_{i_y,y}} && (\delta'(y) < \delta(y) \text{ by definition of } y, \text{ and } v_{i_y,y} > 0 \text{ by definition of } i_y) \\
&= u_{i_y}(\delta) && \text{(by (3), since } y \text{ is critical for } i_y \text{ in } \delta)
\end{aligned}$$

so δ' does not dominate δ. Hence, δ is efficient. □

Despite this characterization, the set of efficient distributions fails to be convex,[6] as in the case of linear utilities (see Bogomolnaia et al., 2005).

Corollary 3. *Every equilibrium distribution is efficient.*

An alternative proof of Corollary 3 is obtained by combining Corollary 1 and Lemma 2.

The following lemma shows that every efficient utility vector is generated by at most one distribution.

Lemma 3. *Let δ and δ' be efficient distributions inducing the same utility vector, that is, $u_i(\delta) = u_i(\delta')$ for all $i \in N$. Then, $\delta = \delta'$.*

Proof. By Lemma 2, for each $x \in \text{supp}(\delta)$ there is an agent for whom x is critical. Denote one such agent by i_x. Then,

$$\begin{aligned}
\delta(x) &= v_{i_x,x} \cdot u_{i_x}(\delta) && \text{(by (3), since } x \text{ is critical for } i_x\text{)} \\
&= v_{i_x,x} \cdot u_{i_x}(\delta') && \text{(by the lemma assumption)} \\
&\le \delta'(x) && \text{(by definition of Leontief utilities).}
\end{aligned}$$

The same inequality $\delta(x) \le \delta'(x)$ trivially holds also for all $x \notin \text{supp}(\delta)$. Since both distributions sum up to C_N, this implies $\delta = \delta'$. □

Consequently, an efficient distribution rule essentially maps a profile to a utility vector.

4.3. Existence, uniqueness, and computation

One common way to obtain an efficient distribution is to maximize a welfare function. Formally, for any strictly increasing function g on $\mathbb{R}_{\ge 0}$, we say that a distribution δ is *g-welfare-maximizing* if it maximizes the weighted sum $\sum_{i \in N} C_i \cdot g(u_i(\delta))$. Clearly, any such distribution is efficient. Whenever g is strictly concave, there is a *unique* g-welfare-maximizing distribution; the straightforward proof is given in Appendix A.

We focus on the special case in which g is the logarithm function. The *Nash welfare* of a distribution δ is defined as the sum of logarithms of the agents' utilities, weighted by their contributions:

$$\mathit{Nash}(\delta) := \sum_{i \in N} C_i \cdot \log u_i(\delta).$$

The *Nash rule* selects a distribution δ that maximizes $\mathit{Nash}(\cdot)$ or, equivalently, the weighted product of the agents' utilities $\prod_{i \in N} u_i^{C_i}$ (with the convention that $0 \log 0 = 0$ and $0^0 = 1$). The following lemmas show that a distribution is in equilibrium if and only if it maximizes Nash welfare. This implies the existence and uniqueness of equilibrium distributions.[7]

[6] Consider an example with three charities $\{a, b, c\}$ and two agents with $v_{1,c} = v_{2,a} = 0$ and $v_{i,x} = 1$ otherwise, and $C_1 = C_2 = 1$. Then, $\delta = (1, 1, 0)$ and $\delta' = (0, 1, 1)$ are both efficient distributions, but not $0.5\,\delta + 0.5\,\delta' = (0.5, 1, 0.5)$.

[7] An alternative *non-constructive* existence proof can be obtained by leveraging a theorem by Rosen (1965), who shows the existence of pure equilibria in n-player games with bounded, closed, and convex strategy sets when the payoff function is continuous in strategy profiles and concave in individual strategies.

Lemma 4. *Every distribution that maximizes Nash welfare is in equilibrium.*

One way to prove Lemma 4 is to analyze the KKT conditions of the constrained maximization problem corresponding to maximizing Nash welfare. Below, we give a more intuitive proof, which helps to illustrate the "social" aspect of the equilibrium distribution. We first show that, in any non-equilibrium distribution, there is a set of agents who "waste" some of their contribution on charities that are only critical for other agents.

Lemma 5. *If δ is an efficient distribution that is not in equilibrium, then N can be partitioned into two disjoint groups of agents, N_+ and $N_- = N \setminus N_+$, such that*

$$\delta(T_{\delta,N_-}) < C_{N_-}; \tag{5}$$

$$\delta(T_{\delta,N_+} \setminus T_{\delta,N_-}) > C_{N_+}. \tag{6}$$

Proof. Let $(\delta_i)_{i\in N}$ be any decomposition of δ. Construct a directed graph G in which the nodes correspond to agents, and there is an arc $i \to j$ if and only if $\delta_i(T_{\delta,j}) > 0$, that is, agent i contributes to a critical charity of j. We call the arc $i \to j$ *strong* if $\delta_i(T_{\delta,j} \setminus T_{\delta,i}) > 0$, that is, agent i contributes to a charity that is critical for j but not for i. Otherwise, we call the arc $i \to j$ *weak*. Since δ is not in equilibrium, by Lemma 1, there is an agent, say agent 1, who contributes to a charity $x \notin T_{\delta,1}$. Since δ is efficient, by Lemma 2, x is critical to some other agent, say agent 2, so G contains a strong arc $1 \to 2$.

If the strong arc is a part of a directed cycle, then we can move a sufficiently small amount ε along the cycle without changing δ. In detail, suppose without loss of generality that the cycle is $1 \to 2 \to \cdots \to k \to 1$, where the involved charities are $x_1 \in T_{\delta,1},\ x_2 \in T_{\delta,2} \setminus T_{\delta,1},\ x_3 \in T_{\delta,3},\ x_4 \in T_{\delta,4},\ \ldots,\ x_k \in T_{\delta,k}$. We assume that x_2 is in $T_{\delta,2} \setminus T_{\delta,1}$ since the arc $1 \to 2$ is strong; in particular, x_2 must be different than x_1. The other arcs may be strong or weak, and some of the x_i may coincide. For every $i \in \{1, \ldots, k-1\}$, move a small amount $\varepsilon > 0$ from $\delta_i(x_{i+1})$ to $\delta_i(x_i)$; move the same ε from $\delta_k(x_1)$ to $\delta_k(x_k)$. Note that the decomposition changes, but the total δ remains the same. Increase ε until one arc of the cycle disappears, or the strong arc becomes weak. Repeat this cycle-removal procedure until all strong arcs are not part of any directed cycle. This process is guaranteed to terminate since in each cycle removal, either the respective strong arc becomes weak or the cycle it is part of is removed. Furthermore, no new (strong) arcs are created as agents do not contribute to additional charities, and the overall distribution δ together with the set of critical charities does not change.

Let G be the graph of the resulting decomposition. Since the total distribution is still δ, which is efficient but not in equilibrium, G still has at least one strong arc, say $j \to k$. Let N_+ be the set of agents accessible from k via a directed path (where $k \in N_+$), and let $N_- := N \setminus N_+$. Since $j \to k$ is not part of any directed cycle, $j \in N_-$.

Due to the strong arc $j \to k$, agents of N_- waste some of their own contributions on critical charities of N_+, that are not critical for themselves. Moreover, the critical charities of N_- do not receive any donations from agents of N_+, since they are not accessible from N_+. This proves (5).

In contrast, the agents in N_+ spend all their contributions on their own critical charities, that are not critical charities of agents outside N_+. In addition, they receive some donations from agents of N_-. This proves (6). □

Proof of Lemma 4. Let δ be an efficient non-equilibrium distribution. We prove that δ is not Nash-optimal.

Let N_- and N_+ be the subsets of agents defined in Lemma 5. If $\delta(T_{\delta,N_-}) = 0$, then $Nash(\delta) = -\infty$ and δ is clearly not Nash-optimal, so we may assume that $\delta(T_{\delta,N_-}) > 0$. We construct a new distribution δ' in the following way.

- Remove a small amount ε from $\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})$, such that each charity loses proportionally to its current distribution. That is, for each charity $x \in T_{\delta,N_+} \setminus T_{\delta,N_-}$, the new distribution is $\delta'(x) := \delta(x) \cdot [1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$.
- Add this ε to $\delta(T_{\delta,N_-})$ such that each charity gains proportionally to its current distribution. That is, for each charity $y \in T_{\delta,N_-}$, the new distribution is $\delta'(y) := \delta(y) \cdot [1 + \varepsilon/\delta(T_{\delta,N_-})]$.

Choose ε sufficiently small such that the sets of critical charities of agents in N_- do not change (that is, no new charities become critical for them). This redistribution has the following effect on the agents' utilities:

- The utility of each agent $i \in N_+$ may decrease by a factor of up to $[1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$. Therefore, the contribution to Nash welfare changes by at least $\Delta_{N_+}(\varepsilon) := C_{N_+} \cdot \log[1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$. We have $\lim_{\varepsilon \to 0} \Delta_{N_+}(\varepsilon)/\varepsilon = -C_{N_+}/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})$, which is larger than -1 by inequality (6).
- The utility of each agent $i \in N_-$ increases by a factor of $[1 + \varepsilon/\delta(T_{\delta,N_-})]$. Therefore, the contribution to Nash welfare increases by $\Delta_{N_-}(\varepsilon) := C_{N_-} \cdot \log[1 + \varepsilon/\delta(T_{\delta,N_-})]$. We have $\lim_{\varepsilon \to 0} \Delta_{N_-}(\varepsilon)/\varepsilon = C_{N_-}/\delta(T_{\delta,N_-})$, which is larger than 1 by inequality (5).

The overall difference in Nash welfare is $\Delta(\varepsilon) := \Delta_{N_+}(\varepsilon) + \Delta_{N_-}(\varepsilon)$, and we have $\lim_{\varepsilon \to 0} \Delta(\varepsilon)/\varepsilon > -1 + 1 = 0$, so $\Delta(\varepsilon) > 0$ for sufficiently small ε. Therefore, $Nash(\delta') > Nash(\delta)$, so δ was not Nash-optimal, completing the proof. □

Lemma 6. *Every equilibrium distribution maximizes Nash welfare.*

Proof. Let δ^* be an equilibrium distribution. For any distribution δ, we derive an upper bound for $Nash(\delta)$ in terms of δ^*. We show that this upper bound is maximized when $\delta = \delta^*$ and is equal to $Nash(\delta)$ for $\delta = \delta^*$. Thus, $Nash(\delta) \leq Nash(\delta^*)$ so δ^* maximizes the Nash welfare.

Formally, let $(\delta^*_i)_{i \in N}$ be any decomposition of δ^* satisfying Lemma 1, and let $N_{\delta^*,x} := \{i \colon x \in T_{\delta^*,i}\}$ be the set of agents for whom x is critical in δ^*. For every distribution δ with $Nash(\delta) > -\infty$, we have

$$Nash(\delta) = \sum_{i \in N} C_i \log(u_i(\delta))$$

$$
\begin{aligned}
&= \sum_{i\in N}\left(\sum_{x\in T_{\delta^*,i}} \delta_i^*(x)\right)\log(u_i(\delta)) && \text{(by (4))}\\
&\le \sum_{i\in N}\sum_{x\in T_{\delta^*,i}} \delta_i^*(x)\cdot\log\left(\frac{\delta(x)}{v_{i,x}}\right)\\
&= \sum_{x\in A}\left(\sum_{i\in N_{\delta^*,x}} \delta_i^*(x)\cdot\log\left(\frac{\delta(x)}{v_{i,x}}\right)\right)\\
&= \sum_{x\in A}\left(\sum_{i\in N_{\delta^*,x}} \delta_i^*(x)\right)\log(\delta(x)) - \sum_{x\in A}\left(\sum_{i\in N_{\delta^*,x}} \delta_i^*(x)\log(v_{i,x})\right)\\
&= \sum_{x\in A}\delta^*(x)\log(\delta(x)) - \sum_{x\in A}\left(\sum_{i\in N_{\delta^*,x}} \delta_i^*(x)\log(v_{i,x})\right). && \text{(by (1))}
\end{aligned}
$$

We claim that, for every fixed δ^*, the latter expression is maximized for $\delta = \delta^*$. The second term is independent of δ. As for the first term $\sum_{x\in A}\delta^*(x)\log(\delta(x))$, consider the optimization problem of maximizing $\sum_{x\in A}\delta^*(x)\log(\delta(x))$ subject to $\sum_{x\in A}\delta(x) = \sum_{x\in A}\delta^*(x)$ (note that δ^* is a constant in this problem). Its Lagrangian is

$$
\sum_{x\in A}\delta^*(x)\log(\delta(x)) + \lambda\cdot\left(\sum_{x\in A}\delta^*(x) - \sum_{x\in A}\delta(x)\right)
$$

Setting the derivative with respect to $\delta(x)$ to 0 gives $\delta^*(x)/\delta(x) = \lambda$ for all $x \in A$. Since $\sum_{x\in A}\delta(x) = \sum_{x\in A}\delta^*(x)$, we must have $\lambda = 1$, so $\delta = \delta^*$. This means that

$$
\mathit{Nash}(\delta) \le \sum_{x\in A}\delta^*(x)\log(\delta^*(x)) - \mathrm{const}(\delta^*).
$$

For $\mathit{Nash}(\delta^*)$, the same derivation holds, but the inequality becomes an equality, since in equilibrium, $\delta_i^*(x) > 0$ only if $u_i(\delta^*) = \delta^*(x)/v_{i,x}$. Therefore,

$$
\mathit{Nash}(\delta) \le \mathit{Nash}(\delta^*),
$$

so δ^* is Nash-optimal. □

Since the logarithm function is strictly concave, Lemma 14 in Appendix A implies that there is a unique distribution that maximizes Nash welfare. Hence, Lemmas 4 and 6 entail that there is a unique equilibrium distribution, and it is efficient, as claimed in Theorem 1.

The equilibrium distribution maximizes a weighted sum of logarithms and can thus be approximated arbitrarily well by considering the corresponding convex optimization problem. For linear utilities, Brandl et al. (2022) show that it is impossible to compute the Nash-optimal distribution exactly, even for binary valuations, since this distribution may involve irrational numbers. By contrast, for Leontief utilities the Nash-optimal distribution is rational whenever the agents' valuations and contributions are rational. This is the case because, given the sets of critical charities for each agent, the equilibrium distribution can be computed using linear programming.[8] In the special case of binary weights, the equilibrium distribution can be computed using a polynomial number of linear programs; see Section 7.

4.4. Cobb-Douglas utilities

It turns out that the equilibrium distribution is unaffected if the agents' Leontief utility functions are replaced with Cobb-Douglas utility functions for the same values for charities. Given values for charities, the Cobb-Douglas utility function of agent i is defined as $u_i(\delta) = \prod_{x\in A} \delta(x)^{v_{i,x}}$. Both Cobb-Douglas and Leontief utility functions belong to the class of utility functions with *constant elasticity of substitution* (see, e.g., Varian, 1992; Mas-Colell et al., 1995). Similarly to the case of maximizing the Nash product of individual utilities, maximizing a single Cobb-Douglas utility function is equivalent to maximizing $\sum_{x\in A} v_{i,x} \cdot \log(\delta(x))$.

Proposition 1. *Given values $(v_{i,x})_{i\in N, x\in A}$, a distribution is in equilibrium for Leontief utility functions if and only if it is in equilibrium for Cobb-Douglas utility functions.*[9]

Proof. We show that Lemma 1 (with the same definition of critical charities) also holds for Cobb-Douglas utilities $u_i(\delta) = \sum_{x\in A} v_{i,x} \cdot \log(\delta(x))$. Looking at the derivative ${}^{du_i(\delta)}/_{d\delta(x)} = v_{i,x}/\delta(x)$, we observe that increasing the amount of contribution of $\arg\max_{x\in A} v_{i,x}/\delta(x) = \arg\min_{x\in A} v_{i,x} \cdot \log(\delta(x))$ gives agent i the largest marginal gains.

⇒: Suppose that, in every decomposition of δ, some agent i contributes to a charity $y \notin T_{\delta,i}$. Fix a decomposition $(\delta_i)_{i\in N}$ of δ. By assumption, $v_{i,y} \cdot \log(\delta(y)) > v_{i,x} \cdot \log(\delta(x))$ for any $x \in T_{\delta,i}$. Agent i can move a sufficiently small amount ε from $\delta_i(y)$ to a charity $x \in T_{\delta,i}$, resulting in a new individual distribution δ_i^ε; this strictly increases her utility as

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{u_i(\delta - \delta_i + \delta_i^\varepsilon) - u_i(\delta)}{\varepsilon}\\
=&\lim_{\varepsilon\to 0} v_{i,x} \cdot \frac{\log(\delta(x)+\varepsilon) - \log(\delta(x))}{\varepsilon} + v_{i,y} \cdot \frac{\log(\delta(y)-\varepsilon) - \log(\delta(y))}{\varepsilon}\\
=&\frac{v_{i,x}}{\delta(x)} - \frac{v_{i,y}}{\delta(y)} > 0.
\end{aligned}
$$

Therefore, δ is not an equilibrium distribution.

⇐: Suppose δ has a decomposition in which each agent i only contributes to charities in $T_{\delta,i}$. In every other strategy of agent i, she must contribute ε less to at least one such

[8] In Fisher markets with linear utility functions, there exists a rational-valued equilibrium distribution, which can be computed *exactly* in polynomial time via the ellipsoid algorithm and Diophantine approximation (Jain, 2007; Vazirani, 2012). Similar techniques can also be used in our setting to give a pseudo-polynomial time algorithm, see Appendix D.

[9] Proposition 1 also holds when the weighted logarithms in the additive representation of Cobb-Douglas utility functions are replaced with an arbitrary strictly concave, strictly increasing function.

charity, $y \in T_{\delta,i}$. Similar to the other direction, it can be shown that $v_{i,y} \cdot (\log(\delta(y)) - \log(\delta(y) - \varepsilon)) > v_{i,x} \cdot (\log(\delta(x) + \varepsilon) - \log(\delta(x)))$ for any charity x. Therefore, the utility of agent i is smaller than $u_i(\delta)$ and the deviation is not beneficial. □

As a consequence, existence and uniqueness of equilibrium distributions carry over to Cobb-Douglas utility functions. However, efficiency breaks down.

Remark 1. The equilibrium distribution can violate efficiency for Cobb-Douglas utilities. Consider two agents with $C_1 = C_2 = 6$ and three charities a, b, and c such that Agent 1 assigns value 1 to charities a and b and Agent 2 assigns value 1 to charities b and c. The other values are 0. Then the equilibrium distribution $\delta^* = (4, 4, 4)$ results in utilities $u_1(\delta^*) = u_2(\delta^*) = 4 \cdot 4 = 16$. However, the distribution $\delta = (3, 6, 3)$ provides more Cobb-Douglas utility to both agents: $u_1(\delta) = u_2(\delta) = 3 \cdot 6 = 18$.

4.5. Public good markets

While equilibrium distributions are defined as Nash equilibria in a strategic game, they can also be seen as market equilibria in a stylized public good market. To this end, consider a market in which each charity corresponds to a divisible public good in unlimited supply. Each agent has a budget of C_i to spend on public goods and preferences on how the endowment C_N should be distributed among the goods. Each public good is available at the same unit price. A distribution is in equilibrium if no agent can increase her utility by redistributing her budget.

Alternatively, equilibrium distributions can be interpreted in a hypothetical public good market with personalized prices as introduced by Lindahl (1919). The following definition is adapted from Definition 5 by Fain et al. (2016). A distribution δ is a *Lindahl equilibrium* if there exist personal price functions $p_1, \ldots, p_n \in \mathbb{R}^A_{\geq 0}$ such that the following two conditions hold.

1. For every agent $i \in N$, δ maximizes $u_i(y)$ subject to $\sum_{x \in A} p_i(x)y(x) \leq C_i$ and $y \in \mathbb{R}^A_{\geq 0}$. That is, y is the best bundle that i could "buy" using her share of the budget.

2. δ maximizes the expression $\sum_{x \in A} y(x) \left(\sum_{i \in N} p_i(x) - 1\right)$ subject to $y \in \mathbb{R}^A_{\geq 0}$, which represents the net profit of a hypothetical producer who collects all agents' payments and "produces" every charity at unit cost.

Fain et al. (2016) and Gul and Pesendorfer (2022) have established relationships between Nash welfare and Lindahl equilibrium for certain types of utility functions. For example, Fain et al. (2016, Theorem 2) show that, if all agents have the same budget C_N/n and all utilities are *scalar-separable* and *non-satiating*, then a Lindahl equilibrium can be computed efficiently by maximizing a concave potential function that generalizes Nash welfare. We prove an analogous result for Leontief utilities (which are neither scalar-separable nor non-satiating) and non-uniform budgets.

Proposition 2. *A distribution is in equilibrium if and only if it is a Lindahl equilibrium distribution.*

Note that, since the equilibrium distribution is unique, the proposition also implies that the Lindahl equilibrium is unique (up to normalization).

Proof. We first show that every equilibrium distribution is a Lindahl equilibrium distribution. Let δ^* be the equilibrium distribution with equilibrium decomposition $(\delta^*_i)_{i\in N}$. We can assume without loss of generality that every charity is valuable to at least one agent, and therefore $\delta^*(x) > 0$ for all $x \in A$. The prices are defined by letting

$$p_i(x) := \frac{\delta^*_i(x)}{\delta^*(x)}$$

for all $i \in N$ and $x \in A$. Then, $\sum_{x\in A} p_i(x)\delta^*(x) = \sum_{x\in A} \delta^*_i(x) = C_i$. By Lemma 1, $p_i(x) = 0$ for every project x not critical for i. Therefore, any other $y \in \mathbb{R}^A_{\geq 0}$ with $\sum_{x\in A} p_i(x)y(x) \leq C_i$ must allocate a smaller amount to some project critical for i, and hence yield a smaller utility for i. So the first part of the definition holds.

For the second part, note that, the profit of *any* $y \in \mathbb{R}^A_{\geq 0}$ for the given prices is

$$\begin{aligned}
\sum_{x\in A} y(x)\left(\sum_{i\in N} p_i(x) - 1\right) &= \sum_{x\in A} y(x)\left(\sum_{i\in N} \frac{\delta^*_i(x)}{\delta^*(x)} - 1\right)\\
&= \sum_{x\in A} y(x)\left(\frac{\delta^*(x)}{\delta^*(x)} - 1\right)\\
&= 0,
\end{aligned}$$

so δ^* vacuously maximizes the profit. Hence, δ^* is a Lindahl equilibrium as claimed.

We now show that every Lindahl equilibrium is an equilibrium distribution. Let z be a Lindahl equilibrium with prices $p_1, \ldots, p_n \in \mathbb{R}^A_{\geq 0}$. Again, we can assume without loss of generality that every charity is valuable to at least one agent. Hence, $z(x) > 0$ for all $x \in A$; otherwise, at least one agent would receive utility 0 and condition 1 of Lindahl equilibrium would not hold for that agent. Since we consider Lindahl equilibrium up to normalization, we can assume without loss of generality $\sum_{x\in A} z(x) = C_N$ and write $\delta = z$ in the following.

First, by the second condition, the expression $g_x := \sum_{i\in N} p_i(x)$ is the same for all $x \in A$. Otherwise, the net profit could be improved by moving funds from charities with low g_x to charities with high g_x. Furthermore, taking the sum of $\sum_{x\in A} p_i(x)\delta(x) \leq C_i$ over all agents in the first condition shows that $g_x \leq 1$.

Second, $p_i(x) = 0$ for $x \notin T_{\delta,i}$; otherwise, as $\delta(x) > 0$, agent i spends on x a positive amount $p_i(x)\delta(x)$, and could use it on her critical charities instead to improve her utility.

Third, $\sum_{x\in A} p_i(x)\delta(x) = C_i$ for every agent i, otherwise she could afford buying a larger "amount" of her critical charities.

We claim that for any group of agents $N_- \subset N$, $C_{N_-} \leq \delta\left(T_{\delta,N_-}\right)$. To see this, note that

$$C_{N_-} = \sum_{i\in N_-} C_i = \sum_{i\in N_-}\sum_{x\in A} p_i(x)\delta(x) \qquad \left(\text{as } C_i = \sum_{x\in A} p_i(x)\delta(x)\right)$$

$$
\begin{aligned}
&= \sum_{i \in N_-} \sum_{x \in T_{N_-,\delta}} p_i(x)\delta(x) && (\text{as } p_i(x) = 0 \text{ for } x \notin T_{\delta,i}) \\
&= \sum_{x \in T_{N_-,\delta}} \sum_{i \in N_-} p_i(x)\delta(x) \\
&\le \sum_{x \in T_{N_-,\delta}} \delta(x) && (\text{as } \sum_{i \in N_-} p_i(x) \le 1 \text{ for all } x) \\
&= \delta\left(T_{\delta,N_-}\right).
\end{aligned}
$$

By (5) in Lemma 5, δ is the equilibrium distribution. □

For general utility functions, these two notions of equilibrium do not coincide, e.g., in Remark 1 for Cobb-Douglas utilities, δ constitutes the Lindahl equilibrium, which means that the Lindahl equilibrium Pareto-dominates the equilibrium distribution.

5. The equilibrium distribution rule

Based on Theorem 1, we define the *equilibrium distribution rule (EDR)* as the distribution rule that, for each profile, returns the unique equilibrium distribution for this profile. In this section, we investigate axiomatic properties of *EDR*.

5.1. Strategyproofness

A distribution rule is *group-strategyproof* if no coalition of agents can gain utility by misreporting their valuations or contributing less. This incentivizes truthful reports and allows for a correct estimation of agents' utilities under different distributions. Furthermore, a group-strategyproof rule ensures that every agent donates the maximal possible contribution, thereby guaranteeing maximal gains from coordination.

Definition 4 (Group-strategyproofness)**.** Given a distribution rule f, a profile P, and a group $G \subseteq N$, a profile P' is called a *manipulation of P by G* if $C'_G \le C_G$ (the contribution of G may decrease), and the valuations of agents in G may change, while the contributions and valuations of all agents in $N \setminus G$ remain the same. Such a manipulation is called *successful* if $u_j(f(P')) \ge u_j(f(P))$ for all $j \in G$ and $u_i(f(P')) > u_i(f(P))$ for at least one $i \in G$, where $(u_i)_{i \in N}$ refers to the utilities in P.

A distribution rule f is *group-strategyproof* if in any profile, no group of agents has a successful manipulation.

We prove that *EDR* is group-strategyproof by leveraging the following lemma.

Lemma 7. *Let δ^1 and δ^2 be two distributions, and $i \in N$ an agent.*

(a) If $u_i(\delta^2) \ge u_i(\delta^1)$, then every charity in $T_{\delta^1,i}$ receives at least as much funding in δ^2, that is: $\delta^2(y) \ge \delta^1(y)$ for all $y \in T_{\delta^1,i}$.

(b) Similarly, if $u_i(\delta^2) > u_i(\delta^1)$, then $\delta^2(y) > \delta^1(y)$ for all $y \in T_{\delta^1,i}$.

Proof. For (a), for every charity $y \in T_{\delta^1,i}$, we have

$$\begin{aligned}
\delta^1(y) &= v_{i,y} \cdot u_i(\delta^1) && \text{(by (3), as } y \text{ is critical for } i \text{ in } \delta^1\text{)} \\
&\leq v_{i,y} \cdot u_i(\delta^2) && \text{(by assumption)} \\
&= v_{i,y} \cdot \min_{x \in A_i} \frac{\delta^2(x)}{v_{i,x}} && \text{(by definition of Leontief utilities)} \\
&\leq v_{i,y} \cdot \frac{\delta^2(y)}{v_{i,y}} && \text{(since } y \in T_{\delta^1,i} \subseteq A_i\text{)} \\
&= \delta^2(y).
\end{aligned}$$

For (b), the first inequality becomes strict. □

Theorem 2. *EDR is group-strategyproof.*

Proof. Suppose by contradiction that some group of agents has a successful manipulation, and let $G \subseteq N$ be an inclusion-maximal such group. For an arbitrary profile P, denote by P' the profile after a successful manipulation by G and by δ^P and $\delta^{P'}$ the respective equilibrium distributions. Since the manipulation succeeds, $u_j(\delta^{P'}) \geq u_j(\delta^P)$ for all $j \in G$ and $u_i(\delta^{P'}) > u_i(\delta^P)$ for at least one $i \in G$. By Lemma 7, $\delta^{P'}(x) \geq \delta^P(x)$ for every charity x that belongs to $T_{\delta^P,j}$ for some $j \in G$, and $\delta^{P'}(x) > \delta^P(x)$ for every charity x in $T_{\delta^P,i}$. This implies

$$\delta^{P'}\left(\bigcup_{j \in G} T_{\delta^P,j}\right) > \delta^P\left(\bigcup_{j \in G} T_{\delta^P,j}\right). \tag{7}$$

We write both equilibrium distributions as decompositions $\delta^P = \sum_{i \in N} \delta_i^P$ and $\delta^{P'} = \sum_{i \in N} \delta_i^{P'}$ satisfying Lemma 1. Since $C'_G \leq C_G$, inequality (7) above must hold for the individual distribution of at least one agent $k \in N \setminus G$, that is,

$$\delta_k^{P'}\left(\cup_{j \in G} T_{\delta^P,j}\right) > \delta_k^P\left(\cup_{j \in G} T_{\delta^P,j}\right).$$

Consequently, at least one charity $x_G \in \cup_{j \in G} T_{\delta^P,j}$ has $\delta_k^{P'}(x_G) > \delta_k^P(x_G)$. By Lemma 1, x_G must be critical for k in $\delta^{P'}$. Therefore,

$$\begin{aligned}
v_{k,x_G} \cdot u_k(\delta^{P'}) &= \delta^{P'}(x_G) && \text{(by (3), as } x_G \text{ is critical for } k \text{ in } \delta^{P'}\text{)} \\
&\geq \delta^P(x_G) && \text{(by Lemma 7, as } x_G \in T_{\delta^P,j} \text{ for some } j \in G\text{).} \\
&\geq v_{k,x_G} \cdot u_k(\delta^P) && \text{(by Leontief utilities),}
\end{aligned}$$

so agent k's utility is not decreased by the group's manipulation. Consequently, k could be added to G—contradicting the maximality of G.

We conclude that no group of agents has a successful manipulation and thus *EDR* is group-strategyproof. □

In fact, the above proof shows that if the total contribution C_G decreases, then the utility of at least one agent in G has to *strictly* decrease under *EDR* since $\sum_{i\in G}\delta_i^{P'}\left(\bigcup_{j\in G}T_{\delta^P,j}\right) < \sum_{i\in G}\delta_i^{P}\left(\bigcup_{j\in G}T_{\delta^P,j}\right)$ and the above argument applies. In particular, an agent receives *strictly* more utility when she increases her contribution. The interpretation of *EDR* as the Nash product rule even allows us to give an explicit lower bound on the utility gain when increasing one's contribution.

Theorem 3. *Under EDR, agents are strictly better off by increasing their contribution.*

Proof. Let P' be the profile where, compared to P, one agent j increased her contribution by $Z > 0$. Let $\delta^P \in \Delta(C)$ and $\delta^{P'} \in \Delta(C+Z)$ be the respective equilibrium distributions.

We claim that $\frac{u_j(\delta^{P'})}{u_j(\delta^P)} \geq \frac{C+Z}{C}$. To see this, define $\delta' = \frac{C+Z}{C}\cdot\delta^P$ and $\delta'' = \frac{C}{C+Z}\cdot\delta^{P'}$ such that $\delta' \in \Delta(C+Z)$ and $\delta'' \in \Delta(C)$. Denote by $NASH_P(\delta)$ the weighted product of agents' utilities in profile P and distribution δ (the exponent of the Nash welfare as previously defined). Then,

$$
\begin{aligned}
1 &\leq \frac{NASH_{P'}(\delta^{P'})}{NASH_{P'}(\delta')} && \text{(by maximality of } \delta^{P'} \text{ in } \Delta(C+Z))\\
&= \frac{NASH_P(\delta^{P'})}{NASH_P(\delta')}\cdot\frac{u_j(\delta^{P'})^Z}{u_j(\delta')^Z} && \text{(as agent } j \text{ increased contribution by } Z)\\
&= \left(\frac{C+Z}{C}\right)^C\cdot\frac{NASH_P(\delta'')}{NASH_P(\delta')}\cdot\frac{u_j(\delta^{P'})^Z}{u_j(\delta')^Z} && \left(\text{as } \delta^{P'} = \frac{C+Z}{C}\cdot\delta''\right)\\
&= \frac{NASH_P(\delta'')}{NASH_P(\delta^P)}\cdot\frac{u_j(\delta^{P'})^Z}{u_j(\delta')^Z} && \left(\text{as } \delta' = \frac{C+Z}{C}\cdot\delta^P\right)\\
&\leq \frac{u_j(\delta^{P'})^Z}{u_j(\delta')^Z} && \text{(by maximality of } \delta^P \text{ in } \Delta(C))
\end{aligned}
$$

Thus, $u_j(\delta^{P'}) \geq u_j(\delta') = \frac{C+Z}{C}\cdot u_j(\delta^P)$ and the auxiliary claim is proved.

If $u_j(\delta^P) > 0$, then the auxiliary claim implies that $u_j(\delta^{P'}) > u_j(\delta^P)$. Otherwise, $u_j(\delta^P) = 0$ implies $C_j = 0$, and $C_j + Z > 0$ implies $u_j(\delta^{P'}) > 0$ by the equilibrium property, so again $u_j(\delta^{P'}) > u_j(\delta^P)$. □

We are not aware of other settings in which the Nash product rule is strategyproof. Theorem 2 strongly relies on Leontief preferences. If, for example, preferences are given by Cobb-Douglas utility functions, *EDR* can be manipulated. To see this, consider the example given in Remark 1. If the first agent only reports a positive value for charity a, her utility increases from $4\cdot 4 = 16$ to $6\cdot 3 = 18$.

5.2. Preference-monotonicity

An important property from the perspective of charity managers is *preference-monotonicity*, which requires that for every agent i and charity $x \in A$, $\delta(x)$ weakly increases when $v_{i,x}$ increases. In other words, a charity can only receive more donations when it becomes more popular.

Definition 5 (Preference-monotonicity)**.** A distribution rule f satisfies *preference-monotonicity* if for every two profiles P and P' which are identical except that $v'_{i,x} > v_{i,x}$ for one agent i and one charity x, we have $f(P')(x) \geq f(P)(x)$.

For linear utilities, strategyproofness implies preference-monotonicity (Brandl et al., 2021). This does not hold for Leontief utilities, even when valuations are binary. Nevertheless, we still have the following.

Theorem 4. *EDR satisfies preference-monotonicity.*

Proof. Let P be a profile and P' a modified profile where one agent i increases her valuation for one charity x (that is, $v'_{i,x} > v_{i,x}$ and $v'_{i,y} = v_{i,y}$ for all $y \in A \setminus x$). Let δ^P and $\delta^{P'}$ be the respective equilibrium distributions. We need to show that $\delta^{P'}(x) \geq \delta^P(x)$.

Let u_i and u'_i be agent i's Leontief utility functions in the two profiles.

By definition of Leontief utilities, $u'_i(\delta^P) = \min(u_i(\delta^P), \delta^P(x)/v'_{i,x})$. We consider two cases, depending on which of the two expressions within the minimum is larger.

Case 1: $u_i(\delta^P) < \delta^P(x)/v'_{i,x}$. Then $u'_i(\delta^P) = u_i(\delta^P)$, and all charities in $T_{\delta^P,i}$ remain critical for i in the new profile. Therefore, by Lemma 1, δ^P is still an equilibrium distribution for P'. By uniqueness of the equilibrium distribution, $\delta^{P'}(x) = \delta^P(x)$.

Case 2: $u_i(\delta^P) \geq \delta^P(x)/v'_{i,x}$. By definition of Leontief utilities,

$$\frac{\delta^{P'}(x)}{v'_{i,x}} \geq u'_i(\delta^{P'}).$$

By strategyproofness (Theorem 2),

$$u'_i(\delta^{P'}) \ \geq \ u'_i(\delta^P).$$

By definition of Leontief utilities,

$$u'_i(\delta^P) = \min\left(u_i(\delta^P),\ \frac{\delta^P(x)}{v'_{i,x}}\right) = \frac{\delta^P(x)}{v'_{i,x}},$$

since by assumption $u_i(\delta^P) \geq \delta^P(x)/v'_{i,x}$. Combining these three inequalities yields $\delta^{P'}(x) \geq \delta^P(x)$, as desired. □

To complement Theorem 4, we observe some other effects of increasing the valuation $v_{i,x}$ of one agent i for one charity x:

- The equilibrium Nash welfare cannot increase; otherwise, the equilibrium distribution in the new profile would also have a higher Nash welfare than the equilibrium distribution of the original profile, with respect to the original valuations; but this contradicts Lemma 6. However, the equilibrium Nash welfare might remain constant if x is not among the critical charities of agent i.

- Similarly, the utility of agent i under the equilibrium distribution cannot increase: if agent i's utility with the new valuation is larger under the new equilibrium, this implies that her utility with the original valuation is also larger in the new equilibrium and thus, there exists a beneficial manipulation (reporting exactly that new valuation instead). This would contradict strategyproofness of *EDR* (Theorem 2). However, agent i's utility might remain constant if x is not among her critical charities.

5.3. Contribution-monotonicity

For some applications, it is desirable if increased contributions do not result in the redistribution of funds that have already been allocated. For example, if agents arrive over time or increase their contributions over time, ideally the mechanism only needs to take care of the additional contributions. This would allow a deployment of the mechanism as an incremental process in which charities can make immediate use of the donations they receive. We formalize this property in the following definition.

Definition 6 (Contribution-monotonicity)**.** A distribution rule f satisfies *contribution-monotonicity* if for every two profiles P and P' where P' can be obtained from P by increasing the contribution of one agent (possibly from 0), $f(P')(x) \geq f(P)(x)$ for all charities $x \in A$.

Theorem 5. *EDR satisfies contribution-monotonicity.*

Proof. Let P and P' be profiles as in Definition 6, so that $C_i' \geq C_i$ for all $i \in N$.

Let δ and δ' be the equilibrium distributions corresponding to profiles P and P', respectively. Fix decompositions of δ and δ' into individual distributions satisfying Lemma 1.

Let A^- , $A^=$, and A^+ be the sets of all charities $x \in A$ with $\delta'(x) < \delta(x)$, $\delta'(x) = \delta(x)$, and $\delta'(x) > \delta(x)$, respectively. Assume for contradiction that A^- is not empty. Thus, $\sum_{i\in N} \delta_i'(A^-) < \sum_{i\in N} \delta_i(A^-)$, so there must be an agent $i \in N$ with $\delta_i'(A^-) < \delta_i(A^-)$, and a charity $y \in A^-$ with $\delta_i'(y) < \delta_i(y)$. But $\delta_i'(A) = C_i' \geq C_i = \delta_i(A)$, so $\delta_i'(A^= \cup A^+) > \delta_i(A^= \cup A^+)$, so there must be a charity $z \in A^= \cup A^+$ with $\delta_i'(z) > \delta_i(z) \geq 0$. By Lemma 1, charities z and y are critical for i under δ' and δ, respectively. This, in particular, implies that $v_{i,z} > 0$ and $v_{i,y} > 0$. Therefore,

$$\frac{\delta'(z)}{v_{i,z}} \leq \frac{\delta'(y)}{v_{i,y}} < \frac{\delta(y)}{v_{i,y}} \leq \frac{\delta(z)}{v_{i,z}},$$

where the first and last inequalities follow from the definition of critical charities. This implies $\delta'(z) < \delta(z)$, a contradiction to $z \in A^= \cup A^+$. □

Remark 2. Theorem 5 yields an alternative proof of the uniqueness of equilibrium distributions, which does not rely on the equivalence with Nash welfare optimality. If δ and δ' are equilibrium distributions for the same profile, then both $\delta'(x) \geq \delta(x)$ and $\delta(x) \geq \delta'(x)$ must hold for every charity $x \in A$, which implies $\delta' = \delta$.

6. Spending dynamics converging to equilibrium

Thus far, we have assumed the existence of a central authority that collects the preferences of all agents and then either distributes the endowment among the charities or recommends to each agent how to distribute her individual contribution. In this section, we show that equilibrium distributions can also be attained in multi-round processes without a central authority, simply by letting agents spend their contribution one after another in a myopically optimal way. Agents need not reveal their preferences explicitly, but they have to be able to observe the donations made in previous rounds.

To this end, we consider infinite processes in which agents repeatedly play best responses against the strategies of the other agents in previous rounds. We first analyze a redistribution dynamics where the endowment remains fixed and agents can redistribute their contribution whenever it is their turn. It turns out that the distribution converges to the equilibrium distribution under a very mild condition on the sequence of agents. We then consider a continuous spending dynamics in which there is constant flow of contributions from each agent (for example, when each donor i has set aside a monthly budget C_i to spend on charitable activities). We focus on the case of round-robin sequences and show that the relative overall distribution (or, equivalently, the average distribution over all rounds) converges to the equilibrium distribution when agents can only observe the distribution given by the last $n-1$ rounds.[10]

These convergence results can be leveraged to make statements in more flexible settings where the set of participating agents, as well as their preferences and contributions, can change over time. The finite number of donations that have been made up to a certain point will always be outweighed by the infinite number of donations that follow. Hence, even with occasional changes to the profile, the relative overall distribution keeps converging towards an equilibrium distribution of the current profile.

6.1. Redistribution dynamics

Let us first consider a dynamics in which the endowment remains fixed and agents repeatedly redistribute their contributions after observing the current overall distribution.

Formally, denote by δ^* the equilibrium distribution and by δ^t the distribution at round t (along with its associated decomposition), e.g., δ^0 equals the null vector as no agent $i \in N$ has yet distributed her contribution C_i. In each round t, allow one agent i_t to (re-)distribute her entire contribution in such a way that her utility is maximized for the new distribution δ^{t+1}, i.e.,

$$\delta_{i_t}^{best} := \arg \max_{\delta_{i_t} \in \Delta(C_{i_t})} u_{i_t} \left(\delta_{i_t} + \sum_{j \neq i_t} \delta_j^t \right);$$
$$\delta^{t+1} := \delta_{i_t}^{best} + \sum_{j \neq i_t} \delta_j^t.$$

[10] The formal statement is stronger as not only the relative overall distribution, but also the distribution given by the last n rounds, converges to the equilibrium distribution.

Lemma 8. *For every round t and agent i_t, there is a unique best response $\delta_{i_t}^{best}$.*

Proof. Since a best response corresponds to a solution of a maximization problem over the closed and bounded set of possible distributions $\delta_{i_t} + \sum_{j \neq i_t} \delta_j^t$ with the continuous objective function u_{i_t}, existence is guaranteed.

To show uniqueness, observe that for the distribution in round $t+1$ (which for simplified notation we denote by $\delta := \delta^{t+1}$), we have $\delta_{i_t}(T_{\delta,i_t}) = C_{i_t}$, that is, agent i_t distributes all her contribution on her critical charities in δ. In any other response δ'_{i_t}, agent i_t must contribute less to at least one charity of T_{δ,i_t}. Therefore, her utility must be lower than $u_{i_t}(\delta)$, so δ'_{i_t} cannot be a best response. □

Before turning to the main result on the convergence of the dynamics, consider the instance given in the introduction as an example. Suppose Donor 2 is the first in the sequence. Her best response, given the initial distribution $(0, 0, 0, 0)$, is to split her donation of \$100 between C and D, so the distribution becomes $(0, 0, 50, 50)$. Next, Donor 1 plays a best response, which splits the donation of \$900 unequally, giving $316.\overline{6}$ to A, $316.\overline{6}$ to B and $266.\overline{6}$ to C. The distribution becomes $(316.\overline{6}, 316.\overline{6}, 316.\overline{6}, 50)$. Then, Donor 2 plays a best response, which moves all her donation to D. The distribution becomes $(316.\overline{6}, 316.\overline{6}, 266.\overline{6}, 100)$. Finally, Donor 1 plays a best response, which moves $16.\overline{6}$ from each of A and B to C. The distribution then becomes $(300, 300, 300, 100)$, which equals the equilibrium distribution. In this particular case, the equilibrium distribution is attained after a finite number of rounds, but in general we can only prove convergence at the limit.

Theorem 6. *Given a profile P, let $\mathcal{S} = (i_0, i_1, i_2, \dots)$ be an infinite sequence of agents updating their individual distributions via best responses such that there is a bound $K \in \mathbb{N}$ on the maximal number of rounds an agent has to wait until she is allowed to redistribute. Then, the redistribution dynamics converges to the equilibrium distribution, i.e., $\lim_{t\to\infty} \delta^t = \delta^*$.*

The proof will proceed in two steps: First, we will show that the amount an arbitrary agent wants to redistribute converges to 0. Then, we will conclude that this can only be the case if the dynamics converges to the equilibrium distribution. All proofs of auxiliary lemmas are deferred to Appendix B.

For the first step, we define a real-valued function Φ on the set of strategy-vectors, such that, whenever some agent deviates to a best response, Φ strictly increases.[11]

[11] The definition of Φ is inspired by the definition of *ordinal potential functions*, which were originally introduced to prove the existence of pure strategy Nash equilibria in congestion games (Rosenthal, 1973; Monderer and Shapley, 1996) and have since then been widely used to prove convergence to equilibrium (e.g., Milchtaich, 1996, 2000, 2004). However, an ordinal potential function increases whenever a player plays a better response, whereas our Φ increases only when a player plays a best response. In fact, the game we consider does not admit an ordinal potential function, as there can be cycles of better responses.

Voorneveld (2000) defines a *best-response potential function*, which is maximized whenever a player plays a best response; our Φ increases when a player plays a best response, but we do not know if it is maximized.

Note that Nash welfare need *not* monotonically increase for best-response sequences.

$$\Phi(\delta_1, \ldots, \delta_n) := \sum_{i \in N} \sum_{x \in A_i} \delta_i(x) \log\left(\frac{v_{i,x}}{\delta(x)}\right) \tag{8}$$

Note that Φ is well-defined, as $\delta(x) = 0$ implies $\delta_i(x) = 0$ for all $i \in N$, and $x \in A_i$ implies $v_{i,x} > 0$.

Lemma 9. *For any best-response sequence $\mathcal{S}$, it holds that $\Phi(\delta^{t+1}) > \Phi(\delta^t)$ for all t.*

The potential Φ is bounded on $\Delta(C_N)$, since

$$\begin{aligned}
\Phi(\delta_1, \ldots, \delta_n) &= \sum_{i \in N} \sum_{x \in A_i} \delta_i(x) \log\left(\frac{v_{i,x}}{\delta(x)}\right) \\
&= \sum_{i \in N} \sum_{x \in A_i} \delta_i(x) \log(v_{i,x}) - \sum_{x \in A} \delta(x) \log(\delta(x)) \\
&\leq \sum_{i \in N} \sum_{x \in A_i} \delta_i(x) \log(v_{i,x}) + \frac{m}{\mathrm{e}} < \infty.
\end{aligned}$$

Therefore, the sequence $(\Phi(\delta^t))_{t \in \mathbb{N}}$ has to converge to some limit. We denote this limit by ϕ^*.

We will now show that the amount an arbitrary agent wants to redistribute converges to 0. By assumption, there exists a round $T \leq K$ by which all agents have already appeared at least once in $\mathcal{S}$. It is sufficient to prove the theorem for the subsequence starting at T. Therefore, from now on, we assume without loss of generality that at round $t = 0$, all agents have already appeared at least once in $\mathcal{S}$, and thus, have contributed the entire amount C_i.

Denote the amount of shifted contributions in round t by c_t:

$$c_t := \frac{1}{2} \|\delta^t - \delta^{t+1}\|_1.$$

When moving from δ^t to δ^{t+1} in round t, agent i_t redistributes c_t from a set of charities $A^-_{i_t}$ to another set $A^+_{i_t}$ with $A^+_{i_t} \cap A^-_{i_t} = \emptyset$. Since the agent is only allowed to redistribute her individual distribution, $c_t \leq \delta^t_{i_t}(A^-_{i_t})$. Furthermore, since she redistributes according to her best response, she gives money only to charities that are critical to her in the new distribution, so $\delta^{t+1}_{i_t}(x) = 0$ for all $x \in A$ with $\delta^{t+1}(x)/v_{i_t,x} > u_{i_t}(\delta^{t+1})$ and $u_{i_t}(\delta^{t+1}) = \delta^{t+1}(x^+)/v_{i_t,x^+}$ for every $x^+ \in A^+_{i_t}$. An illustrative example is given in Figure 1. In particular, $\delta^{t+1}(x^-)/v_{i_t,x^-} \geq u_{i_t}(\delta^{t+1}) = \delta^{t+1}(x^+)/v_{i_t,x^+}$ for all $x^- \in A^-_{i_t}$ and $x^+ \in A^+_{i_t}$.

Define $d_i(\delta)$ as the amount of contribution that would be shifted by an agent i if the current distribution (along with its associated decomposition) were δ and it was her turn to respond. Note that we define $d_i(\delta)$ for all agents, not only the one who actually plays her best response; in particular, $d_{i_t}(\delta^t) = c_t$ for all t. Note also that δ is the equilibrium distribution if and only if $d_i(\delta) = 0$ for all $i \in N$.

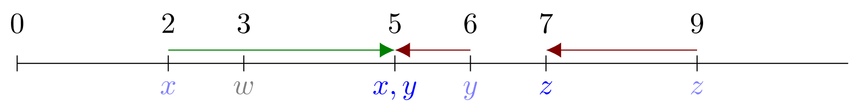

Figure 1: An instance with four charities (named w, x, y, z), $\delta^t = (3, 2, 6, 9)$, and an agent i_t with $\delta^t_{i_t} = (0, 2, 2, 2)$ and Leontief utilities with binary weights $v_{i_t} = (0, 1, 1, 1)$. Then, $\delta^{best}_{i_t} = (0, 5, 1, 0)$, $\delta^{t+1} = (3, 5, 5, 7)$, $c_t = 3$, $A^-_{i_t} = \{y, z\}$, $A^+_{i_t} = \{x\}$.

Lemma 10. *For any sequence $\mathcal{S}$, round $t \geq 0$, and agent $j \in N$,*

$$d_j(\delta^t) \leq d_{i_t}(\delta^t) + d_j(\delta^{t+1}).$$

Intuitively, the lemma can be seen as a "triangle inequality": the left-hand side denotes the direct distance from δ^t towards j's optimal redistribution; the right-hand side denotes the distance along an indirect path that first goes to δ^{t+1} and then proceeds from there towards j's optimal redistribution.

For any agent $j \in N$ and round t, we know that j will get the chance to redistribute her contribution in at most K rounds by assumption. Denote this next round by $t' \leq t + K$. So,

$$\begin{aligned}
\sum_{\ell=t}^{t'} c_\ell &= \sum_{\ell=t}^{t'} d_{i_\ell}(\delta^\ell) \\
&\geq \sum_{\ell=t}^{t'} \Big(d_j(\delta^\ell) - d_j(\delta^{\ell+1}) \Big) && \text{(by Lemma 10)} \\
&= d_j(\delta^t) - d_j(\delta^{t'+1}) \\
&= d_j(\delta^t) && \text{(as } d_j(\delta^{t'+1}) = 0 \text{ after agent } j\text{'s best response).}
\end{aligned}$$

Thus, we have an upper bound on the maximum amount any agent would like to shift at any given round t.

Corollary 4. *For all rounds t, $\sum_{\ell=t}^{t+K} c_\ell \geq \max_{i \in N} d_i(\delta^t)$.*

We combine this with Lemma 9 to show that the amount an agent wants to redistribute converges to 0.

Lemma 11. *For any sequence $\mathcal{S}$ and agent $j \in N$, $\lim_{t\to\infty} d_j(\delta^t) = 0$.*

We can now complete the proof of Theorem 6.

Proof of Theorem 6. For any $\mathcal{S}$, since $(\delta^t)_{t\in\mathbb{N}}$ is an infinite sequence in the closed set of distributions in the (bounded) simplex $\Delta(C_N)$, the Bolzano-Weierstrass theorem states that it has a convergent subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$ with limit $\delta \in \Delta(C_N)$. Furthermore, by Lemma 11, $\lim_{k\to\infty} d_i(\delta^{t_k}) = 0$ for *any* convergent subsequence, implying $d_i(\lim_{t\to\infty} \delta^{t_k}) = 0$ for every agent $i \in N$, and so $\lim_{t\to\infty} \delta^{t_k} = \delta^*$ for every convergent subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$. Thus, $\lim_{t\to\infty} \delta^t = \delta^*$. □

Remark 3. For binary Leontief utilities, the potential simplifies to $\Phi(\delta_1, \dots, \delta_n) = -\sum_{x\in A} \delta(x)\log(\delta(x))$, i.e., lexicographic improvements of δ increase the potential. Consequently, $\widehat{\Phi}(\delta) := \sum|\delta(x) - \delta(y)|$, where the sum is taken over all (unordered) pairs of distinct charities $x, y \in A$, is an alternative potential. Moreover, $\widehat{\Phi}$ has the advantage that the potential increases linearly (not only quadratically) in the redistributed amounts. It can then be shown that Theorem 6 holds for *any* sequence $\mathcal{S}$ in which each agent appears infinitely often.

6.2. Round-robin spending dynamics

Let us now move on to a model in which there is constant flow of donations and each agent repeatedly donates her contribution C_i when it is her turn. To this end, fix some order of the agents (say, $1, 2, \dots, n$) and denote by $\delta_i^t(x)$ the total amount of contributions of agent i to charity x until round t. At each round $t \geq 0$, agent $i_t = 1 + (t \mod n)$ donates C_{i_t} in such a way that her utility is maximized with respect to the previous donation of each other agent, i.e.,

$$\begin{aligned}
\delta_t^{best} &:= \arg\max_{\delta_{i_t}\in\Delta(C_{i_t})} u_{i_t}\left(\delta_{i_t} + \sum_{t-n<s<t} \delta_s^{best}\right);\\
\delta^{t+1} &:= \delta^t + \delta_t^{best};\\
\delta_{i_t}^{t+1} &:= \delta_{i_t}^t + \delta_t^{best},
\end{aligned}$$

where the distribution of the contribution of agent i_t in round t is denoted by δ_t^{best}.[12]

Lemma 8 still applies: each agent's best response is unique. To compare δ^t with the equilibrium distribution δ^* (where each agent only contributed once), we scale δ_i^t by the number of donations of agent i until round t, which equals $\lfloor (t + n - i)/n \rfloor$.

To illustrate the process, consider the example from the introduction for the sequence $(2, 1, 2, 1, \dots)$. First, Donor 2 splits her donation of \$100 between C and D, resulting in $\delta^1 = (0, 0, 50, 50)$. Next, Donor 1 plays a best response, which splits the donation of \$900 unequally, giving $316.\overline{6}$ to A, $316.\overline{6}$ to B and $266.\overline{6}$ to C leading to $\delta^2 = (316.\overline{6}, 316.\overline{6}, 316.\overline{6}, 50)$. Then, Donor 2 donates another \$100 to D under her best response. The overall distribution becomes $\delta^3 = (316.\overline{6}, 316.\overline{6}, 316.\overline{6}, 150)$. It is straightforward to see that from now on, Donor 1 will always split her contribution equally on A, B, and C whereas Donor 2 will only donate to D. Thus, $\lim_{t\to\infty} 2\delta_1^t/t = (300, 300, 300, 0)$ and $\lim_{t\to\infty} 2\delta_2^t/t = (0, 0, 0, 100)$, showing convergence to the equilibrium distribution.

[12] We here assume that the "observation window" of each agent is given by the last $n-1$ rounds. Computer simulations suggest that convergence also holds for larger observation windows.

Theorem 7. *Given a profile P, the continuous round-robin spending dynamics converges to the equilibrium distribution, i.e.,*

$$\lim_{t\to\infty}\sum_{i\in N}\frac{1}{\lfloor (t+n-i)/n\rfloor}\delta_i^t=\delta^*.$$

Proof. For every t, note that δ_t^{best} is the same distribution as the best response of agent i_t under the redistribution dynamics of Section 6.1 with round-robin sequence $\mathcal{S}$. Thus, Theorem 6 implies that the sum of the last n individual distributions (one per agent) converges to the equilibrium distribution, i.e., $\lim_{t\to\infty}\sum_{k=t-n+1}^{t}\delta_k^{best}=\delta^*$. Consequently, for t being a multiple of n, the sum

$$\sum_{i\in N}\frac{1}{\lfloor (t+n-i)/n\rfloor}\delta_i^t=\sum_{i\in N}\frac{n}{t}\delta_i^t=\frac{n}{t}\sum_{\ell=1}^{t/n}\sum_{k=(\ell-1)n}^{\ell n-1}\delta_k^{best}$$

converges to δ^* as $t\to\infty$. As for arbitrarily large t not being a multiple of n, donations from rounds $\lfloor n/t\rfloor,\dots,t-1$ do only have an arbitrarily small impact on $\sum_{i\in N}\frac{1}{\lfloor (t+n-i)/n\rfloor}\delta_i^t$, convergence to δ^* holds for the whole sequence. □

Remark 4. In the spirit of Proposition 1, the convergence results for binary weights also apply to Cobb-Douglas utility functions, as not only equilibrium distributions but also best responses coincide.

7. Leontief utilities with binary weights

In this section, we consider the special case of binary Leontief weights, i.e., $v_{i,x}\in\{0,1\}$ for all agents $i\in N$ and charities $x\in A$. Equivalently, each agent i has a non-empty set of *approved charities* $A_i\subseteq A$ and her utility from a distribution δ is

$$u_i(\delta)=\min_{x\in A_i}\delta(x).$$

For each charity $x\in A$, we denote by $N_x\subseteq N$ the set of agents who approve charity x. For a subset of agents $N'\subseteq N$, we denote $A_{N'}:=\bigcup_{i\in N'}A_i$ as the set of charities approved by at least one member of N'. Note that, for every charity $x\in A$ and every agent $i\in N_x$,

$$\delta(x)\geq u_i(\delta). \tag{9}$$

Binary weights allow for further insights into the structure of the equilibrium distribution, which in turn yield new interpretations and additional properties of *EDR*.

For linear utilities with binary weights, a distribution is in equilibrium if and only if each agent contributes only to charities she approves. Brandl et al. (2021) refer to this axiom as *decomposability*.

Definition 7 (Decomposable distribution)**.** Given a profile with binary weights ($v_{i,x} \in \{0,1\}$), a distribution δ is *decomposable* if it has a decomposition $(\delta_i)_{i\in N}$ such that $\delta_i(x) = 0$ for every charity $x \notin A_i$. Equivalently, it has a decomposition satisfying the following, instead of (2):

$$\sum_{x\in A_i} \delta_i(x) = C_i \qquad \text{for all } i \in N. \tag{10}$$

The equivalence of decomposable distributions and equilibrium distributions no longer holds with Leontief utilities: there are decomposable distributions that are not in equilibrium even when there is only one agent. Nevertheless, decomposability can be used to establish two appealing alternative interpretations of *EDR* for binary weights.

7.1. Egalitarianism for charities

Motivated by Example 1, we aim at a rule which distributes money on the charities as equally as possible while still respecting the preferences of the donors. One rule that comes to mind selects a distribution that, among all decomposable distributions, maximizes the smallest amount allocated to a charity. Subject to this, it maximizes the second-smallest allocation to a charity, and so on. We define it formally using the *leximin* relation.

Definition 8. Given two vectors $\mathbf{x}, \mathbf{y}$ of the same size, we say that $\mathbf{x}$ *is leximin-higher than* $\mathbf{y}$ (denoted $\mathbf{x} \succ_{lex} \mathbf{y}$) if the smallest value in $\mathbf{x}$ is larger than the smallest value in $\mathbf{y}$; or the smallest values are equal, and the second-smallest value in $\mathbf{x}$ is larger than the second-smallest value in $\mathbf{y}$; and so on. $\mathbf{x} \succeq_{lex} \mathbf{y}$ means that either $\mathbf{x} \succ_{lex} \mathbf{y}$ or the multiset of values in $\mathbf{x}$ is the same as that in $\mathbf{y}$.

Definition 9. The *charity egalitarian rule* selects a distribution δ^* that, among all decomposable distributions, maximizes the distribution vector by the leximin order, that is: $\delta^* \succeq_{lex} \delta$ for every decomposable distribution δ.

The leximin order on the closed and convex set of decomposable distributions is connected, every two vectors are comparable, and there exists a unique maximal element (otherwise, any convex combination of two different maximal elements would be leximin-higher than the maximal elements). Therefore, the charity egalitarian rule selects a unique distribution and is well-defined. We prove below that the returned distribution is the equilibrium distribution, resulting in an alternative characterization of *EDR* for binary weights.

Theorem 8. *With binary weights, the charity egalitarian rule and EDR are equivalent.*

Proof. By uniqueness of the equilibrium distribution (Theorem 1), it is sufficient to show that the charity egalitarian distribution is in equilibrium. Let δ^{CHEG} be the decomposable charity egalitarian distribution, with decomposition $\delta^{CHEG} = \sum_{i\in N} \delta_i^{CHEG}$. Suppose for contradiction that δ^{CHEG} is not in equilibrium. By Lemma 1, there is an agent $i \in N$ who contributes to a non-critical charity $x \in A_i$, that is, $\delta_i^{CHEG}(x) > 0$

and $\delta^{CHEG}(x) > u_i(\delta^{CHEG})$. Let $y \in A_i$ be a critical charity of agent i, that is, $\delta^{CHEG}(y) = u_i(\delta^{CHEG})$.

If agent i now moves $1/2(\delta^{CHEG}(x) - \delta^{CHEG}(y))$ from x to y, the resulting distribution is still decomposable, as both x and y are in A_i. It is leximin-higher than δ^{CHEG}, contradicting the leximin-maximality of δ^{CHEG}. □

Remarkably, this new interpretation of *EDR* ignores the Leontief utilities of the agents and does not directly take into account the different contributions. Instead, they enter indirectly through the constraints induced by decomposability.

Theorem 8 implies that *EDR* can be computed by solving the following program, with variables δ_x for all $x \in A$ and $\delta_{i,x}$ for all $i \in N, x \in A$:

$$\begin{aligned}
&\operatorname{lex\,max\,min}\{\delta_x\}_{x\in A} && \text{subject to}\\
&\delta_x = \sum_{i\in N} \delta_{i,x} && \text{for all } x \in A\\
&\sum_{x\in A_i} \delta_{i,x} = C_i && \text{for all } i \in N\\
&\delta_{i,x} \geq 0, \delta_x \geq 0 && \text{for all } i \in N, x \in A_i
\end{aligned}$$

where "lex max min" refers to finding a solution vector that is maximal in the leximin order subject to the constraints, and the second constraint represents decomposability. It is well-known that such leximin optimization with k objectives and linear constraints can be solved by a sequence of k linear programs (see, e.g., Ehrgott, 2005, Sect. 5.3).

Corollary 5. *With binary weights, the equilibrium distribution can be computed by solving at most m linear programs.*

7.2. Egalitarianism for agents

While *EDR* is egalitarian from the point of view of the charities, one could also consider a rule that is egalitarian from the point of view of the agents. The *conditional egalitarian rule* aims to balance the agents' utilities without disregarding their approvals. It selects a decomposable distribution that, among all decomposable distributions, maximizes the utility vector by the leximin order, that is: $\mathbf{u}(\delta^{CEG}) \succeq_{lex} \mathbf{u}(\delta)$ for every decomposable distribution δ.

Theorem 9. *With binary weights, the conditional egalitarian rule and EDR are equivalent.*

Proof. By uniqueness of the equilibrium distribution (Theorem 1), it is sufficient to show that every conditional egalitarian distribution is in equilibrium. Let δ^{CEG} be a conditional egalitarian distribution with decomposition $\delta^{CEG} = \sum_{i\in N} \delta_i^{CEG}$. Suppose for contradiction that δ^{CEG} is not in equilibrium. Then, some agent $i \in N$ contributes to a non-critical charity $x \in A_i$, that is, $\delta_i^{CEG}(x) > 0$ and $\delta^{CEG}(x) > u_i(\delta^{CEG})$.

Let $D := \min\left(\delta_i^{CEG}(x),\ \delta^{CEG}(x) - u_i(\delta^{CEG})\right)$; our assumptions imply that $D > 0$. Construct a new distribution δ' from δ^{CEG} by changing only δ_i^{CEG}: remove D from charity x, and add $D/|A_i|$ to every charity in A_i (including x). The utility of i increases by $D/|A_i|$, since:

- $\delta'(x) = \delta^{CEG}(x) - D + D/|A_i| \geq u_i(\delta^{CEG}) + D/|A_i|$ by definition of D;
- $\delta'(y) = \delta^{CEG}(y) + D/|A_i| \geq u_i(\delta^{CEG}) + D/|A_i|$ for all $y \in A_i \setminus x$, by (9) with equality for $y \in T_{\delta^{CEG},i}$.
- So $u_i(\delta') = \min(\delta'(x), \min_{y \in A_i \setminus x} \delta'(y)) = u_i(\delta^{CEG}) + D/|A_i| > u_i(\delta^{CEG})$.

Moreover, if the utility of some agent j decreases—that is, $u_j(\delta') < u_j(\delta^{CEG})$—then this must be because of the decrease in the distribution to x, so x must be a critical charity for agent j in δ', i.e., $u_j(\delta') = \delta'(x) \geq u_i(\delta') > u_i(\delta^{CEG})$.

Thus, moving from δ^{CEG} to δ', the number of agents with utility larger than $u_i(\delta^{CEG})$ strictly increases, and the utility of each agent with utility at most $u_i(\delta^{CEG})$ in δ^{CEG} does not decrease. Therefore, $\mathbf{u}(\delta') \succ_{lex} \mathbf{u}(\delta^{CEG})$. Since δ' is decomposable, this contradicts the optimality of δ^{CEG}. □

Theorem 9 implies that the equilibrium distribution can be computed by solving the following program, with variables u_i for all $i \in N$ and $\delta_{i,x}$ for all $i \in N, x \in A_i$.

$$
\begin{aligned}
&\operatorname{lex\,max\,min}\{u_i\}_{i\in N} && \text{subject to} \\
&\qquad u_i \leq \delta_{i,x} && \text{for all } i \in N, x \in A_i \\
&\qquad \sum_{x \in A_i} \delta_{i,x} = C_i && \text{for all } i \in N \\
&\qquad \delta_{i,x} \geq 0, u_i \geq 0 && \text{for all } i \in N, x \in A_i
\end{aligned}
$$

Using standard algorithms for lexicographic max-min optimization (see, e.g., Ehrgott, 2005, Sect. 5.3), this program can be solved using at most n linear programs.

Thus, we have three algorithms for computing the equilibrium distribution in the case of binary weights: one requires at most m linear programs; one requires at most n linear programs; and one requires a single convex (non-linear) program. It would be interesting to investigate which of these algorithms is most efficient in practice.

Note that, for general Leontief utilities, equilibrium distributions do *not* necessarily maximize the leximin vector of either the charities or the agents.

Example 3 (For general Leontief utilities, *EDR*, the conditional egalitarian rule, and the charity egalitarian rule are different from one another)**.** There are three charities x, y, z, and two agents, both of whom contribute 30. The values of Agent 1 are $(1, 2, 0)$ and the values of Agent 2 are $(0, 1, 1)$.

The charity egalitarian rule returns the leximin-maximal distribution for charities (subject to decomposability), which is $(20, 20, 20)$ with decomposition $(20, 10, 0)$, $(0, 10, 20)$. It is not in equilibrium, since Agent 1 contributes to charity x, which is not critical.

The conditional egalitarian rule returns the leximin-maximal distribution for agents (subject to decomposability), which is $(15, 30, 15)$, with utility vector $(15, 15)$ and decomposition $(15, 15, 0)$, $(0, 15, 15)$. It is not in equilibrium, since Agent 2 contributes to charity y, which is not critical.

To compute the equilibrium distribution, we can guess that x, y are critical for Agent 1 and y, z are critical for Agent 2, and solve the system of four equations: $\delta(x) = u_1$; $\delta(y) = 2u_1 = u_2$; $\delta(z) = u_2$; $\delta(x) + \delta(y) + \delta(z) = 60$. The solution is $(12, 24, 24)$; Agent 1 contributes $(12, 18, 0)$ and has utility 12, while Agent 2 contributes $(0, 6, 24)$ and has utility 24. One can verify that this distribution is indeed in equilibrium, so it is the equilibrium distribution.

7.3. Welfare functions maximized by *EDR*

Based on the observation that *EDR* coincides with both the Nash product rule and the conditional egalitarian rule for binary weights, a natural question to ask is which other welfare notions are maximized by *EDR* subject to decomposability.

For this, we take a closer look at g-welfare (see Section 4.3 and Appendix A), but this time subject to decomposability. Clearly, every g-welfare-maximizing distribution is efficient. Below we prove that efficiency is retained even when maximizing among decomposable distributions.

Lemma 12. *Let g be any strictly increasing function, and let δ be a distribution that maximizes the g-welfare among all decomposable distributions. Then δ is unique and efficient.*

Proof sketch. Suppose for contradiction that δ is not efficient. By Lemma 2, there is a charity $x \in \text{supp}(\delta)$ which is not critical for any agent. Then, one agent who contributes to x would be able to shift a small amount uniformly to the set of her critical charities such that x is still not critical for any agent. The resulting distribution is still decomposable, and Pareto dominates δ, contradicting the maximality of δ in g-welfare.

Uniqueness is proved similarly to Lemma 14, using the fact that the set of decomposable distributions is convex, i.e., mixing decomposable distributions results in another decomposable distribution. □

Note that uniqueness holds only within the set of decomposable distributions; there might exist non-decomposable distributions with the same g-welfare, as shown in the following example.

Example 4. Let $g(x) = -x^{-1}$ (a strictly increasing function). Suppose there are two agents with $A_1 = \{a\}$, $A_2 = \{b\}$, $C_1 = 2$, $C_2 = 1$. Then, the unique decomposable distribution $\delta^* = (2, 1)$ has the same g-welfare ($-2/2 - 1/1 = -2$) as the non-decomposable distribution $\delta = (1.5, 1.5)$ ($-2/1.5 - 1/1.5 = -2$).

The Nash product rule is often considered a compromise between maximizing utilitarian welfare ($\sum_{i \in N} C_i \cdot u_i$) and egalitarian welfare (maximizing the utility of the agent with smallest utility; notice that the conditional egalitarian rule is a refinement). This can be seen when considering the family of g-welfare functions $\sum_{i \in N} C_i \cdot \text{sgn}(p) \cdot u^p$ for $p \neq 0$ where the limit $p \to 0$ corresponds to $\sum_{i \in N} C_i \cdot \log(u_i)$ and $p \to -\infty$ approaches egalitarian welfare.

The equivalence between conditional egalitarian welfare and Nash welfare extends to a larger class of g-welfare functions. This is shown by the following theorem, proved in Appendix C.1.

Theorem 10. *Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a function that satisfies the following conditions:*

1. *g is strictly increasing on $\mathbb{R}_{\geq 0}$ and differentiable on $\mathbb{R}_{>0}$, and*
2. *$xg'(x)$ is non-increasing on $\mathbb{R}_{>0}$.*

Then, the equilibrium distribution maximizes g-welfare among all decomposable distributions.

Property (1) ensures that social welfare is indeed increasing when an individual's utility increases and small changes in individual utilities only cause small changes in the total social welfare. Property (2) implies that increasing utilities are discounted "at least logarithmically" when being translated to welfare.

In particular, Theorem 10 holds for all g-welfare functions $\sum_{i \in N} C_i \cdot \text{sgn}(p) \cdot u^p$ with $p < 0$. However, it ceases to hold when $p > 0$, as the following proposition (whose proof is deferred to Appendix C.2) shows.

Proposition 3. *For each $p > 0$, maximizing the g-welfare with respect to $g(u) = u^p$ subject to decomposability does not always return the equilibrium distribution.*

Theorem 10 stresses the fact that *EDR* can be motivated not only from a game-theoretic and axiomatic point of view, but also from a welfarist perspective.

8. Discussion

Under the assumption that donors' preferences can be modeled using Leontief utility functions, *EDR* turns out to be an exceptionally attractive rule for funding charitable organizations. It satisfies efficiency, is group-strategyproof, can be computed via convex programming or a pseudo-polynomial time algorithm, and returns the limit of natural spending dynamics. Moreover, in the case of binary weights, *EDR* maximizes a wide range of possible welfare functions and can be computed via linear programming. These results stand in sharp contrast to the previously studied case of linear utilities, where a far-reaching impossibility has shown the incompatibility of efficiency, strategyproofness, and a very weak form of fairness (Brandl et al., 2021). The literature in this stream of research has produced various rules such as the *conditional utilitarian rule*, the *Nash product rule*, the *random priority rule*, or the *sequential utilitarian rule* which trade off these properties against one another (Bogomolnaia et al., 2005; Duddy, 2015; Aziz et al., 2020; Brandl et al., 2021, 2022).

An important question is to which extent our results carry over to other concave utility functions, which offer a natural middle-ground between linear and Leontief utilities. Proposition 1 and Remark 4 show that equilibrium existence and uniqueness as well as convergence of the best-response-based spending dynamics also hold for Cobb-Douglas utilities. However, the equilibrium distribution may fail to be efficient (Remark 1) and *EDR* is manipulable for Cobb-Douglas utilities (see Section 5.1).

Equilibrium distributions can be interpreted as market equilibria for a pure public good market with unlimited supply. This perspective allows interesting comparisons to Fisher markets, arguably the simplest markets for divisible private goods. Equilibria in Fisher markets are connected to Nash welfare maximization under fairly general assumptions about individual utilities, whereas this connection appears to be more volatile in our public good markets. On the other hand, even for Leontief preferences, Fisher market equilibria cannot be computed exactly, and the mechanism that returns the equilibrium is manipulable.

Leontief preferences can be refined by breaking ties between distributions lexicographically, similar to leximin utilities. More precisely, rather than only caring about the minimum of $\delta(x)/v_{i,x}$ for $x \in A$, agents can rank all distributions according to the leximin relation (Definition 8) among the vectors $(\delta(x)/v_{i,x})_{x\in A}$. Remarkably, all of our results for general Leontief valuations carry over to these utility functions by adapting the proofs accordingly. It should be noted, however, that lexicographic Leontief preferences are discontinuous.

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APPENDIX

A. Welfare-maximizing distributions

Let g be a strictly increasing function. The *g-welfare* of a distribution δ is defined as the following weighted sum:

$$g\text{-welfare}(\delta) := \sum_{i \in N} C_i \cdot g(u_i(\delta)).$$

Quantifying welfare enables us to compare and rank all possible utility vectors, which by Lemma 3 induces a social welfare ordering over all distributions $\delta \in \Delta(C_N)$ by g-welfare(δ).

Inversely, every continuous social welfare ordering without any "welfare dependencies" between the agents' utilities can be represented by a g-welfare function; see Chapter 2 in the book by Moulin (1988) for a detailed discussion. Additionally weighting agents by their contributions, we arrive at the very expressive class of g-welfare functions.

A distribution is called *g-welfare-maximizing* if it maximizes the g-welfare, i.e., it always chooses a maximal element of the corresponding social welfare ordering. Clearly, every g-welfare-maximizing distribution is efficient. When g is concave (equivalently: when the induced social welfare ordering satisfies the Pigou-Dalton principle), a g-welfare-maximizing distribution can be found by solving a convex program where the variables are $(u_i)_{i \in N}$ and $(\delta_x)_{x \in A}$:

$$\begin{aligned}
&\text{maximize} \sum_{i \in N} C_i \cdot g(u_i) && \text{subject to} && (11)\\
&\sum_{x \in A} \delta_x \leq C_N && && \\
&u_i \leq \delta_x / v_{i,x} && \text{for all } i \in N, x \in A_i && \\
&u_i \geq 0 && \text{for all } i \in N && \\
&\delta_x \geq 0 && \text{for all } x \in A. &&
\end{aligned}$$

The following technical lemmas prove uniqueness of the welfare-maximizing distribution when g is strictly concave (and strictly increasing).

Lemma 13. *For every strictly concave, strictly increasing function g, every constant $t \in (0,1)$, and every two distributions $\delta \neq \delta'$,*

$$g\text{-welfare}\left(t\delta + (1-t)\delta'\right) \quad > \quad \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta)).$$

Proof. For every agent $i \in N$, by the concavity of the minimum operator,

$$u_i\left(t\delta + (1-t)\delta'\right) \geq t u_i(\delta) + (1-t) u_i(\delta).$$

Therefore,

$$g\text{-welfare}\left(t\delta + (1-t)\delta'\right) \geq \sum_{i \in N} C_i \cdot g\left(t \cdot u_i(\delta) + (1-t) \cdot u_i(\delta')\right)$$

$$
\begin{aligned}
&> t \sum_{i\in N} C_i \cdot g(u_i(\delta)) + (1-t) \sum_{i \in N} C_i \cdot g(u_i(\delta'))\\
&= t \cdot g\text{-welfare}(\delta) + (1-t)\cdot g\text{-welfare}(\delta')\\
&\geq \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta))
\end{aligned}
$$

where the first inequality follows from monotonicity and the second one from strict concavity. □

Lemma 14. *For every strictly concave, strictly increasing function g, there is a unique g-welfare-maximizing distribution.*

Proof. Assume for contradiction that there exist two different g-welfare-maximizing distributions δ and δ'. Since both distributions are efficient, by Lemma 3 they induce two different utility vectors $(u_i(\delta))_{i\in N}$ and $(u_i(\delta'))_{i\in N}$. By Lemma 13, for any $t \in (0,1)$,

$$
\begin{aligned}
g\text{-welfare}\left(t\delta + (1-t)\delta'\right) &> \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta))\\
&= g\text{-welfare}(\delta') = g\text{-welfare}(\delta).
\end{aligned}
$$

This contradicts the assumption that δ and δ' are g-welfare-maximizing. □

B. Proofs of auxiliary lemmas for Theorem 6 (dynamics)

Theorem 6. *Given a profile P, let $\mathcal{S} = (i_0, i_1, i_2, \dots)$ be an infinite sequence of agents updating their individual distributions via best responses such that there is a bound $K \in \mathbb{N}$ on the maximal number of rounds an agent has to wait until she is allowed to redistribute. Then, the redistribution dynamics converges to the equilibrium distribution, i.e., $\lim_{t\to\infty} \delta^t = \delta^*$.*

Recall our definition of a potential function (8):

$$
\Phi(\delta_1, \dots, \delta_n) := \sum_{i\in N}\sum_{x\in A_i} \delta_i(x) \log\left(\frac{v_{i,x}}{\delta(x)}\right).
$$

Lemma 9. *For any best-response sequence $\mathcal{S}$, it holds that $\Phi(\delta^{t+1}) > \Phi(\delta^t)$ for all t.*

Proof. First, observe that an agent's best response going from δ^t to δ^{t+1} can be described by the following continuous process: as long as the agent spends a positive amount on a non-critical charity, transfer money from such a charity to all critical charities equally, until either (i) at least one more charity becomes critical, or (ii) the agent no longer spends a positive amount on a non-critical charity. This process can be interpreted as a sequence of transfers, where each transfer of amount $\varepsilon > 0$ goes from a charity x with higher weighted distribution to a charity y $\left(\frac{\delta(x)}{v_{i,x}} > \frac{\delta(y)}{v_{i,y}}\right)$ such that after the transfer, the weighted distribution of the former charity remains at least as high as that of the latter: $\frac{\delta(x)-\varepsilon}{v_{i,x}} \geq \frac{\delta(y)+\varepsilon}{v_{i,y}}$.

For each t, since the difference between δ^t and δ^{t+1} is caused by transfers, and each amount ε transferred from one charity to another charity causes a change of ε in distribution for both charities, it suffices to prove that each transfer increases the potential, i.e., $\Phi(\delta^\varepsilon) - \Phi(\delta) > 0$ for arbitrary $\delta \in \Delta(C_N)$ and $\varepsilon > 0$ where δ and δ^ε denote the distributions before and after the transfer.

To see this, note that

$$\begin{aligned}
\Phi(\delta^\varepsilon) - \Phi(\delta) = {} & (\delta_i(x) - \varepsilon)\log\left(\frac{v_{i,x}}{\delta(x)-\varepsilon}\right) + (\delta_i(y)+\varepsilon)\log\left(\frac{v_{i,y}}{\delta(y)+\varepsilon}\right) \\
& + \sum_{j\in N\setminus i:\delta_j(x)>0} \delta_j(x)\log\left(\frac{v_{j,x}}{\delta(x)-\varepsilon}\right) + \sum_{j\in N\setminus i:\delta_j(y)>0} \delta_j(y)\log\left(\frac{v_{j,y}}{\delta(y)+\varepsilon}\right) \\
& - \delta_i(x)\log\left(\frac{v_{i,x}}{\delta(x)}\right) - \delta_i(y)\log\left(\frac{v_{i,y}}{\delta(y)}\right) \\
& - \sum_{j\in N\setminus i:\delta_j(x)>0} \delta_j(x)\log\left(\frac{v_{j,x}}{\delta(x)}\right) - \sum_{j\in N\setminus i:\delta_j(y)>0} \delta_j(y)\log\left(\frac{v_{j,y}}{\delta(y)}\right) \\
= {} & \sum_{j\in N:\delta_j(x)>0} \delta_j(x)\log\left(\frac{\delta(x)}{\delta(x)-\varepsilon}\right) + \sum_{j\in N:\delta_j(y)>0} \delta_j(y)\log\left(\frac{\delta(y)}{\delta(y)+\varepsilon}\right) \\
& + \varepsilon\left(\log\left(\frac{v_{i,y}}{\delta(y)+\varepsilon}\right) - \log\left(\frac{v_{i,x}}{\delta(x)-\varepsilon}\right)\right) \\
= {} & \delta(x)\log\left(\frac{\delta(x)}{\delta(x)-\varepsilon}\right) + \delta(y)\log\left(\frac{\delta(y)}{\delta(y)+\varepsilon}\right) \\
& + \varepsilon\left(\log\left(\frac{v_{i,y}}{\delta(y)+\varepsilon}\right) - \log\left(\frac{v_{i,x}}{\delta(x)-\varepsilon}\right)\right) \\
> {} & 0
\end{aligned}$$

as the last term is nonnegative by $\frac{\delta(x)-\varepsilon}{v_{i,x}} \geq \frac{\delta(y)+\varepsilon}{v_{i,y}}$ and the first two terms sum up to something strictly positive which can be seen by using $\log(1+x) > \frac{x}{1+x}$ for $x > -1$ and $x \neq 0$:

$$\begin{aligned}
\delta(x)\log\left(\frac{\delta(x)}{\delta(x)-\varepsilon}\right) + \delta(y)\log\left(\frac{\delta(y)}{\delta(y)+\varepsilon}\right) &> \delta(x)\cdot\frac{\frac{\varepsilon}{\delta(x)-\varepsilon}}{1+\frac{\varepsilon}{\delta(x)-\varepsilon}} + \delta(y)\cdot\frac{\frac{-\varepsilon}{\delta(x)+\varepsilon}}{1+\frac{-\varepsilon}{\delta(x)+\varepsilon}} \\
&= \delta(x)\cdot\frac{\varepsilon}{\delta(x)} + \delta(y)\cdot\frac{-\varepsilon}{\delta(y)} \\
&= 0.
\end{aligned}$$

□

Lemma 10. *For any sequence $\mathcal{S}$, round $t \geq 0$, and agent $j \in N$,*

$$d_j(\delta^t) \leq d_{i_t}(\delta^t) + d_j(\delta^{t+1}).$$

Proof. If $d_j(\delta^t) \leq d_{i_t}(\delta^t)$, the statement holds trivially. Hence, assume that $d_j(\delta^t) > d_{i_t}(\delta^t)$. In particular, $j \neq i_t$.

Let $\tilde{\delta}_j^{t+1}$ and $\tilde{\delta}^{t+1}$ be the (hypothetical) individual distribution of agent j and the overall distribution had she been able to implement her best response at round t.

Denote the sets of charities that would be affected by agent j's best response at δ^t by $A_j^- := \{x^- \in A_j : \tilde{\delta}_j^{t+1}(x^-) < \delta_j^t(x^-)\}$ and $A_j^+ := \{x^+ \in A_j : \tilde{\delta}_j^{t+1}(x^+) > \delta_j^t(x^+)\}$. Then,

$$\frac{\tilde{\delta}^{t+1}(x^-)}{v_{j,x^-}} \geq \frac{\tilde{\delta}^{t+1}(x^+)}{v_{j,x^+}} \text{ for all } x^- \in A_j^- \text{ and } x^+ \in A_j^+\text{; and} \tag{12}$$

$$\tilde{\delta}_j^{t+1}(x) = 0 \text{ for all } x \in A \text{ with } u_j(\tilde{\delta}^{t+1}) < \frac{\tilde{\delta}^{t+1}(x)}{v_{j,x}} \tag{13}$$

hold by definition of best responses.

Now, a lower bound for $d_j(\delta^{t+1})$ is given by the amount shifted from charities in A_j^- under j's best response in round $t+1$. Again, denote by $\tilde{\delta}_j^{t+2}$ and $\tilde{\delta}^{t+2}$ agent j's best response in round $t+1$ and the corresponding overall distribution; note that both (12) and (13) hold also with $t+1$ replaced by $t+2$.

Consider first the special case in which agent i_t did not change her contribution to charities in $A_j^- \cup A_j^+$, that is, $\delta^t(x) = \delta^{t+1}(x)$ for all $x \in A_j^- \cup A_j^+$. If $d_j(\delta^{t+1}) < d_j(\delta^t)$, then a smaller amount is transferred from charities in A_j^- and to charities in A_j^+ in j's best response at δ^{t+1} than in j's best response at δ^t, so by (12), there exist charities $x^- \in A_j^-$ and $x^+ \in A_j^+$ such that $\tilde{\delta}^{t+2}(x^-) > \tilde{\delta}^{t+2}(x^+) \geq v_{j,x^+} \cdot u_i(\tilde{\delta}^{t+2})$ and thus, $\tilde{\delta}_j^{t+2}(x^-) > 0$. This contradicts (13) with $t+2$ instead of $t+1$. Thus, $d_j(\delta^{t+1}) \geq d_j(\delta^t)$ and the claim follows.

Consider now the general case, in which agent i_t may have changed her contribution to some charities in $A_j^- \cup A_j^+$. We claim that the total transfer of i_t and then j (i.e., $d_{i_t}(\delta^t) + d_j(\delta^{t+1})$) cannot be less than the transfer if j were to act alone (i.e., $d_j(\delta^t)$). The reason is similar to the previous paragraph: If this total transfer is less than $d_j(\delta^t)$, then there exist charities $x^- \in A_j^-$ and $x^+ \in A_j^+$ such that $\tilde{\delta}^{t+2}(x^-) > \tilde{\delta}^{t+2}(x^+) \geq v_{j,x^+} \cdot u_j(\tilde{\delta}^{t+2})$ and $\tilde{\delta}_j^{t+2}(x^-) > 0$, which is a contradiction. Hence, $d_{i_t}(\delta^t) + d_j(\delta^{t+1}) \geq d_j(\delta^t)$, as desired. □

Lemma 11. *For any sequence $\mathcal{S}$ and agent $j \in N$, $\lim_{t\to\infty} d_j(\delta^t) = 0$.*

Proof. We prove the equivalent statement: $\lim_{t\to\infty} \max_{i\in N} d_i(\delta^t) = 0$. Assume for contradiction that there exists $\gamma > 0$ such that for all $T > 0$ there exists $T' \geq T$ with $\max_{i\in N} d_i(\delta^{T'}) \geq \gamma$.

Recall that ϕ^* is the limit of the increasing potential $\Phi(\delta^t)$ as $t \to \infty$. Choose some T such that $\phi^* - \Phi(\delta^T) < \frac{\gamma^2}{4C_N K^2(m-1)^2}$ and $T' \geq T$ with $\max_{i\in N} d_i(\delta^{T'}) \geq \gamma$.

By Corollary 4, $\sum_{\ell=T'}^{T'+K} c_\ell \geq \max_{i\in N} d_i(\delta^{T'}) \geq \gamma$. Thus, there exists some $t \in \{T'+1, \ldots, T'+K\}$ with $c_t \geq \gamma/K$. Consequently, in round t, agent i_t transfers at least $\varepsilon = \gamma/(K(m-1))$ from some charity x to some other charity y.

The upper bound on $\log(1+x)$ from Lemma 9 can be refined to $\log(1+x) > \frac{x}{1+x} + \frac{x^2}{(2+x)^2}$ for $x > -1$ and $x \neq 0$, so we get

$$
\begin{aligned}
\Phi(\delta^{t+1}) - \Phi(\delta^{T'}) &\geq \Phi(\delta^{t+1}) - \Phi(\delta^{t}) \\
&> \delta^t(x)\frac{\left(\frac{\varepsilon}{\delta^t(x)-\varepsilon}\right)^2}{\left(2+\frac{\varepsilon}{\delta^t(x)-\varepsilon}\right)^2} + \delta^t(y)\frac{\left(\frac{-\varepsilon}{\delta^t(y)+\varepsilon}\right)^2}{\left(2+\frac{-\varepsilon}{\delta^t(y)+\varepsilon}\right)^2} \\
&= \delta^t(x)\frac{\varepsilon^2}{(2\delta^t(x)-\varepsilon)^2} + \delta^t(y)\frac{\varepsilon^2}{(2\delta^t(y)+\varepsilon)^2} \\
&> \delta^t(x)\frac{\varepsilon^2}{(2\delta^t(x)-\varepsilon)^2} \\
&> \frac{\varepsilon^2}{4\delta^t(x)} > \frac{\varepsilon^2}{4C_N} > \phi^* - \Phi(\delta^T) \\
&> \phi^* - \Phi(\delta^{T'})
\end{aligned}
$$

This implies $\Phi(\delta^{t+1}) > \phi^*$. But this is impossible, since $\Phi(\delta^t)$ is increasing with t and converges to ϕ^*.

Thus, $\lim_{t\to\infty} d_i(\delta^t) = 0$ for every agent i. □

C. Proofs omitted from Section 7

C.1. Proof of Theorem 10

Theorem 10. *Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a function that satisfies the following conditions:*

1. *g is strictly increasing on $\mathbb{R}_{\geq 0}$ and differentiable on $\mathbb{R}_{>0}$, and*
2. *$xg'(x)$ is non-increasing on $\mathbb{R}_{>0}$.*

Then, the equilibrium distribution maximizes g-welfare among all decomposable distributions.

The proof requires some additional definitions and lemmas and proceeds as follows. First, we show that it is sufficient to prove the statement for *reduced* profiles (Definition 12 and Lemma 17), which are profiles in which each agent approves only charities that receive the same amount in the equilibrium distribution. Then we prove that, in any reduced profile, the equilibrium distribution δ^* maximizes g-welfare, not only in the set of decomposable distributions, but even in a larger set of *weakly decomposable* distributions (Definition 11). To do this, we prove that, for any weakly-decomposable distribution $\delta \neq \delta^*$, there exists a modification δ', which is weakly-decomposable but has a higher g-welfare than δ.

Recall that $[z] := \{1, 2, \ldots, z\}$ for each positive integer z.

Definition 10. Given any distribution δ, define $\mathcal{P}(\delta)$ as a partition of the charities into subsets allocated the same amount. That is, $\mathcal{P}(\delta) := (X_1, \ldots, X_p)$ for some integer $p \geq 1$, where $\cup_{k=1}^{p} X_k = A$, and for each $k \in [p]$, all charities in X_k receive the same amount, $\delta(x) = w_k$ for all $x \in X_k$, and the amounts are ordered such that $0 \leq w_1 < \cdots < w_k$.

Note that $w_1 = 0$ if and only if there exist charities that receive no funding.

Lemma 15. *Let δ^* be the equilibrium distribution, and $(X_1^*, \ldots, X_p^*) = \mathcal{P}(\delta^*)$ be its charity partition. For each $k \geq 1$, let N_k^* be the set of agents who approve one or more charities of X_k^*, but do not approve any charity of $\cup_{\ell<k} X_\ell^*$. Then in equilibrium, the agents of N_k^* contribute only to charities of X_k^*, that is:*

$$\begin{aligned}\delta^*(X_k^*) &= C_{N_k^*}, \text{ and}\\ w_k^* &= C_{N_k^*}/|X_k^*| = \delta^*(X_k^*)/|X_k^*|.\end{aligned}$$

Proof. The utility of all agents in N_k^* is w_k^*, so the set of their critical charities is contained in X_k^*. In equilibrium they contribute only to charities in X_k^* by Lemma 1.

All charities in X_k^* receive the same amount, so this amount must be $C_{N_k^*}/|X_k^*|$. □

Note that, if there are charities not approved by any agent (or approved only by agents who contribute 0), then all these charities will be in X_1^*, and we will have $w_1^* = C_{N_1^*} = 0$.

Definition 11. A distribution δ is called *weakly decomposable* if it has a decomposition in which each agent i only contributes to charities x with $\delta^*(x) \geq u_i(\delta^*)$, where δ^* denotes the equilibrium distribution.

With binary weights, $x \in A_i$ implies $\delta^*(x) \geq u_i(\delta^*)$, so every decomposable distribution is weakly decomposable. Therefore, it is sufficient to prove that δ^* maximizes g-welfare among all weakly decomposable distributions.

The set of weakly decomposable distributions is again convex and can be characterized as follows.

Lemma 16. *A distribution δ is weakly decomposable if and only if, for every $\ell \in [p]$,*

$$\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right). \tag{14}$$

Proof. A distribution δ is weakly decomposable if and only if there exists a decomposition of δ where for every $\ell \in [p]$, agents of N_ℓ^* only contribute to charities of $\cup_{k=\ell}^{p} X_k^*$. This holds if and only if $\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \sum_{k=\ell}^{p} C_{N_\ell^*}$ for every $\ell \in [p]$. By Lemma 15, this is equivalent to the condition $\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$ for every $\ell \in [p]$. □

To simplify the proof of Theorem 10, we introduce the following class of profiles.

Definition 12. A profile is called *reduced* if, in its equilibrium distribution δ^*, for every agent i, there exists a $k \in [p]$ such that $A_i \subseteq X_k^*$, that is, all charities approved by an agent belong to the same class in the partition induced by δ^*.

Note that, in a reduced profile, all charities approved by agent i receive in equilibrium the same amount $u_i(\delta^*)$, and therefore are all critical for i, that is, $T_{\delta^*,i} = A_i$ for all $i \in N$.

Lemma 17. *If Theorem 10 is true for reduced profiles, then it is true for all profiles.*

Proof. Let P be any profile, and δ^* its equilibrium distribution. Let P' be its reduced profile where, compared to P, every agent i has removed her approval from every charity x with $\delta^*(x) > u_i(\delta^*)$. Then, δ^* is the equilibrium distribution for P', too (by the same decomposition). By assumption, Theorem 10 is true for P', so δ^* maximizes g-welfare among all distributions that are weakly decomposable with respect to P'. Since the equilibrium distribution is the same in P and P', the set of weakly decomposable distributions is the same too.

The profile P differs from P' by having additional approvals, which could only decrease the maximal possible g-welfare. But δ^* yields the same welfare in P and P'. Therefore, δ^* necessarily maximizes g-welfare among all distributions that are weakly decomposable with respect to P, too. □

Proof of Theorem 10. Based on Lemma 17, we assume without loss of generality that we are given a reduced profile. Let $X_1^*, \ldots, X_p^*$, and $N_1^*, \ldots, N_p^*$ be the partitioning of charities and agents induced by the equilibrium distribution δ^*, and $w_1^* < \cdots < w_p^*$ the corresponding allocations. By Lemma 15, each charity in X_k^* receives $w_k^* = \delta^*(X_k^*)/|X_k^*|$, and every agent $i \in N_k^*$ has utility w_k^*. Since the profile is reduced, $T_{\delta^*,i} = A_i \subseteq X_k^*$ for all $i \in N_k^*$.

Let δ be any weakly decomposable distribution different than δ^*. We prove that δ does not maximize g-welfare among weakly decomposable distributions by showing a modification δ' of δ, which is weakly decomposable but has a higher g-welfare than δ.

Since $\delta \neq \delta^*$ and both distributions sum up to C_N, there must be charities $x^-, x^+ \in A$ with $\delta(x^-) < \delta^*(x^-)$ and $\delta(x^+) > \delta^*(x^+)$, respectively. Consequently, one of the following two cases has to apply:

- If $\delta(X_k^*) = \delta^*(X_k^*)$ for all $k \in [p]$, let $X_r^* = X_s^*$ ($r = s$) be a class that contains a charity x^- with $\delta(x^-) < \delta^*(x^-)$.
- Otherwise, let r be the largest index in $[p]$ for which $\delta(X_r^*) \neq \delta^*(X_r^*)$. Weak decomposability of δ and Lemma 16 imply that $\delta(X_r^*) > \delta^*(X_r^*)$. As $\delta(X_k^*) = \delta^*(X_k^*)$ for all $k > r$, there must be an $s \leq r$ such that there exists a charity x^- in X_s^* with $\delta(x^-) < \delta^*(x^-)$; choose $s \leq r$ to be the largest index with this property.

In both cases, we define $X^- \subseteq X_s^*$ as the set of all charities x in X_s^* with $\delta(x) < \delta^*(x)$, and $X^+ \subseteq X_r^*$ as the set of all charities x in X_r^* with $\delta(x) > \delta^*(x)$; both sets must be non-empty by construction. The case $r > s$ is depicted in Figure 2.

Starting from δ, transfer a sufficiently small amount ε uniformly from X^+ to X^-; call the resulting distribution δ'. We choose ε small enough such that it does not change the order relations between charities inside and outside X^+ and X^-, that is, for all $x^- \in X^-$ and $x^+ \in X^+$: $\delta'(x^+) > \delta'(x)$ for all $x \in A$ with $\delta(x^+) > \delta(x)$,

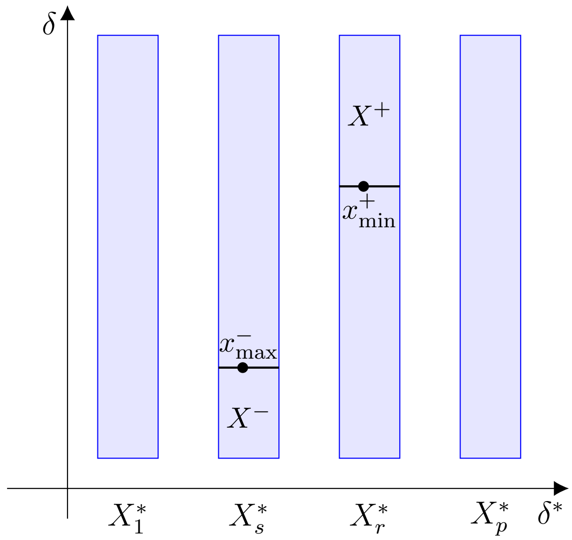

Figure 2: Charity sets in the proof of Theorem 10, for the case $r > s$. The horizontal position of a charity denotes its allocation in δ^*; the vertical position denotes its allocation in δ.

and analogously, $\delta'(x^-) < \delta'(x)$ for all $x \in A$ with $\delta(x^-) < \delta(x)$. In particular, since $\delta(x^+) > \delta^*(x^+) \geq \delta^*(x^-) > \delta(x^-)$, we have $\delta'(x^+) > \delta'(x^-)$.

We claim that δ' is weakly decomposable. By Lemma 16, it suffices to show that (14) holds for δ', that is, $\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$ for every $\ell \in [p]$. Note that $\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) = \delta\left(\cup_{k=\ell}^{p} X_k^*\right)$ for all $\ell \leq s$ and all $\ell \geq r+1$, so for these indices, (14) for δ' follows from the weak-decomposability of δ. It therefore remains to prove (14) for $\ell \in \{s+1, \ldots, r\}$. This set is non-empty only when $s < r$, which is possible only in the second case above.

Our choices of r and s ensure that $\delta\left(\cup_{k=r}^{p} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right)$ and $\delta\left(\cup_{k=1}^{s} X_k^*\right) < \delta^*\left(\cup_{k=1}^{s} X_k^*\right)$. For ε sufficiently small, the same inequalities hold between δ' and δ^*. Moreover, for $s < \ell \leq r$,

$$\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) = \delta'\left(\cup_{k=r}^{p} X_k^*\right) + \delta'\left(\cup_{k=\ell}^{r-1} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right) + \delta\left(\cup_{k=\ell}^{r-1} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$$

where the first inequality holds because $\delta'\left(\cup_{k=r}^{p} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right)$ and $\delta'(X_k^*) = \delta(X_k^*)$ for all $k \notin \{r, s\}$, and the second inequality holds because, for each $k \in \{s+1, \ldots, r-1\}$, all charities x in X_k^* satisfy $\delta(x) \geq \delta^*(x)$ by definition of s. Therefore, by Lemma 16, δ' is still weakly decomposable.

We now analyze the effect of this redistribution on the agents' utilities. For that, we prove an auxiliary claim on critical charities of agents under δ. Define $x^+_{\min} \in \arg\min_{x^+ \in X^+} \delta(x^+)$ as a charity from X^+ with minimal allocation in δ and $x^-_{\max} \in \arg\max_{x^- \in X^-} \delta(x^-)$ as a charity from X^- with maximal contribution in δ.

Claim. For every agent $i \in N$, either $T_{\delta,i} \cap X^- = \emptyset$ or $T_{\delta,i} \subseteq X^-$. Similarly, either $T_{\delta,i} \cap X^+ = \emptyset$ or $T_{\delta,i} \subseteq X^+$.

Proof of claim. We prove the claim for X^-; the proof for X^+ is analogous. By definition of critical charities, $T_{\delta,i} \subseteq A_i$. Since the profile is reduced, A_i is contained in a single partition class. If this partition class is not the one that contains X^-, namely X^*_s, then $T_{\delta,i} \cap X^- = \emptyset$. Otherwise, $T_{\delta,i} \subseteq X^*_s$. Now, if $u_i(\delta) > \delta(x^-_{\max})$, then $\delta(x) > \delta(x^-_{\max})$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \cap X^- = \emptyset$; and if $u_i(\delta) \leq \delta(x^-_{\max})$, then $\delta(x) \leq \delta(x^-_{\max})$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \subseteq X^-$.

Back to proof of theorem. Denote by "losers" the agents who lose utility from the redistribution. The claim implies that all the losers have $T_{\delta,i} \subseteq X^+$; each of them loses $\varepsilon/|X^+|$. Moreover, all losers have $A_i \subseteq X^+$: this is because $A_i \subseteq X^*_r$ (since the profile is reduced), and $\delta(x_A) \geq \delta(x_T) \geq \delta(x^+_{\min})$ for all $x_A \in A_i$ and $x_T \in T_{\delta,i}$. Therefore, in equilibrium, all losers give all their contributions to charities in X^+. This implies that the contributions of all losers sum up to at most $\delta^*(X^+) = w^*_r \cdot |X^+|$. Then, for every loser i,

$$g\left(u_i(\delta)\right) - g\left(u_i(\delta')\right) \leq g\left(\delta(x^+_{\min})\right) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) \tag{15}$$

by concavity of g (which follows from the assumption that $xg'(x)$ is non-increasing).

Denote by "gainers" the agents who gain utility from the redistribution. The claim implies that every agent with $T_{\delta,i} \cap X^- \neq \emptyset$ is a gainer; each of them gains $\varepsilon/|X^-|$. Moreover, every agent with $A_i \cap X^- \neq \emptyset$ is a gainer: this is because $A_i \cap X^- \neq \emptyset$ implies $\delta(x_A) \leq \delta(x^-_{\max})$ for at least one charity $x_A \in A_i$, and $\delta(x_T) \leq \delta(x_A)$ for all charities $x_T \in T_{\delta,i}$. Therefore, in equilibrium, every agent who contributes a positive amount to at least one charity in X^- must be a gainer. So the contributions of all gainers must sum up to at least $\delta^*(X^-) = w^*_s \cdot |X^-|$. Then, for every gainer i,

$$g\left(u_i(\delta')\right) - g\left(u_i(\delta)\right) \geq g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta(x^-_{\max})\right) \tag{16}$$

by concavity of g.

Therefore, by (15) and (16), the increase in g-welfare from δ to δ' is at least

$$\begin{aligned} & w^*_s \cdot |X^-| \cdot \left[g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta(x^-_{\max})\right)\right] \\ & -w^*_r \cdot |X^+| \cdot \left[g\left(\delta(x^+_{\min})\right) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right)\right]. \end{aligned} \tag{17}$$

Since g is strictly concave,

$$g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g(\delta(x^-_{\max})) > \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right);$$

$$g(\delta(x^+_{\min})) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) < \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right).$$

Plugging this into (17), we get that the increase in g-welfare is larger than

$$w^*_s \cdot |X^-| \cdot \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - w^*_r \cdot |X^+| \cdot \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right).$$

By our choice of ε, we have $w^*_r = \delta^*(x^+_{\min}) < \delta(x^+_{\min})$, so $w^*_r < \delta(x^+_{\min}) - \varepsilon/|X^+|$ for sufficiently small ε. Similarly, $w^*_s = \delta^*(x^-_{\max}) > \delta(x^-_{\max}) + \varepsilon/|X^-|$ for sufficiently small ε. Therefore, the increase in g-welfare is larger than

$$\begin{aligned} &\varepsilon \cdot \left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) \\ &-\varepsilon \cdot \left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right). \end{aligned} \tag{18}$$

By our choice of ε, $\delta(x^-_{\max}) + \varepsilon/|X^-| < \delta(x^+_{\min}) - \varepsilon/|X^+|$. By the assumption on g, $xg'(x)$ is non-increasing in x. Therefore, the expression in (18) is at least 0, so the increase in g-welfare from δ to δ' is larger than 0. This means that δ does not maximize g-welfare.

Since δ was any weakly decomposable distribution different than δ^*, we conclude that δ^* maximizes g-welfare subject to weak decomposability in any reduced profile. By Lemma 17, the same is true in any profile. □

C.2. Proof of Proposition 3

Proposition 3. *For each $p > 0$, maximizing the g-welfare with respect to $g(u) = u^p$ subject to decomposability does not always return the equilibrium distribution.*

Proof. For a fixed $p > 0$, consider a profile consisting of two agents with binary weights and approval sets $\{a\}$ and $\{a, b\}$, and respective contributions $C_1 = \max\left(\left(2^{p-1} \cdot p\right)^{-1/p}, 2\right)$ and $C_2 = 1$. Since $C_1 \geq 2$, the equilibrium distribution is $(C_1, 1)$. We claim that the decomposable distribution $(C_1 + 1, 0)$ yields a higher g-welfare, that is,

$$\begin{aligned} &C_1 \cdot g(C_1 + 1) + 1 \cdot g(0) > C_1 \cdot g(C_1) + 1 \cdot g(1) \\ \iff &C_1 \cdot (g(C_1 + 1) - g(C_1)) > 1. \end{aligned}$$

For every $p \geq 1$, g is convex, so

$$\begin{aligned} &g(C_1 + 1) - g(C_1) \geq g'(C_1) \cdot 1 = p \cdot C_1^{p-1} \\ \implies &C_1 \cdot (g(C_1 + 1) - g(C_1)) \geq p \cdot C_1^p \geq p \cdot 2^p \geq 2 > 1. \end{aligned}$$

For every $0 < p < 1$, g is strictly concave, so

$$\begin{aligned} g(C_1 + 1) - g(C_1) > g'(C_1 + 1) \cdot 1 &= p \cdot (C_1 + 1)^{p-1} \\ &> p \cdot (2C_1)^{p-1} \qquad \text{(since } p - 1 < 0 \text{ and } C_1 > 1\text{)} \end{aligned}$$

and so

$$C_1 \cdot (g(C_1+1) - g(C_1)) > 2^{p-1} \cdot p \cdot C_1^p$$
$$\geq 2^{p-1} \cdot p \cdot \left(\left(\frac{1}{2^{p-1}p} \right)^{\frac{1}{p}} \right)^p = 1.$$

In both cases, the equilibrium distribution does not maximize g-welfare. □

D. Efficient computability of equilibrium distributions

The equilibrium distribution can be computed by solving a convex program. In this section, we prove that, in fact, it can be computed *exactly* in pseudo-polynomial time.

Note that the equilibrium distribution does not change when individual valuations are rescaled. Similarly, rescaling contributions preserves the share each charity receives. Thus, for the sake of simplicity, we assume throughout this section that all valuations and contributions are natural numbers. We prove that the equilibrium distribution can be computed in time $\text{poly}(n, m, \log_2(v_{\max}), C_N)$ where $v_{\max} := \max_{i \in N, x \in A} v_{i,x}$. In particular, if all contributions are equal, i.e., $C_i = 1$ for all $i \in N$, then the run-time is polynomial in the binary encoding length of the input.

As a first step, we prove that the equilibrium distribution δ^* and its utility profile (which we denote by u^*) are rational with a bounded binary encoding length.

Lemma 18. *If the agents' valuations $v_{i,x}$ and contributions C_i are natural numbers, then the equilibrium distribution δ^* and its utility profile u^* are rational-valued.*

Moreover, the binary encoding length of (δ^*, u^*) is bounded by a polynomial function of the binary encoding length of $v_{i,x}$ and C_i.

Proof. For each $i \in N$, let T_i be a non-empty set of charities. Consider the following linear program (LP), with variables u_i (for $i \in N$), d_x (for $x \in A$), and $d_{i,x}$ (for $i \in N$ and $x \in A$):

$$
\begin{aligned}
& d_x = u_i \cdot v_{i,x} && \text{for all } i \in N, x \in T_i; \\
& d_x \geq u_i \cdot v_{i,x} && \text{for all } i \in N, x \in A \setminus T_i; \\
& \sum_{x \in T_i} d_{i,x} = C_i && \text{for all } i \in N; \\
& \sum_{i \in N} d_{i,x} = d_x && \text{for all } x \in A; \\
& d_{i,x} \geq 0 && \text{for all } i \in N, x \in A.
\end{aligned}
$$

Every solution of this LP (if any) represents a distribution $\delta(x) = \delta_x$ for all x, with a decomposition $\delta_i(x) = \delta_{i,x}$ for all i, x, such that each agent i contributes only to charities in T_i, and all the charities in T_i are critical for i. By Lemma 1, such a distribution has to coincide with the equilibrium distribution.

The equilibrium distribution δ^* is a solution to the above LP whenever $T_i = T_{\delta^*,i}$ for all $i \in N$. By assumption, the coefficients of this LP are all rational. Therefore, by well-known properties of linear programming, the LP has a rational solution, with binary encoding length bounded by a polynomial function of the representation length of its coefficients.

We can even give an explicit bound on the representation length of δ^*.

Given the equilibrium distribution δ^*, we construct an undirected graph where vertices correspond to charities, and there is an edge between $x, y \in A$ if and only if there exists an agent $i \in N$ with $x, y \in T_{\delta^*,i}$. Each component of that graph can be considered separately, as the sum of contributions to that component equals the sum of contributions of agents for whom parts of the component are critical charities.

Thus, let $A' \subseteq A$ be a subset of m' charities forming a component and $N' \subseteq N$ the subset of agents contributing to charities in A'. Given some $x_1 \in A'$, there needs to be at least one other charity x_2 that can be reached in one step, i.e., there exists an agent $i_1 \in N'$ such that $\delta^*(x_1)/v_{i_1,x_1} = \delta^*(x_2)/v_{i_1,x_2}$. Hence, $\delta^*(x_2) = (v_{i_1,x_2}/v_{i_1,x_1}) \cdot \delta^*(x_1)$. Next, there exists another charity $x_3 \in A'$ that can be reached in one step from either x_1 or x_2. In general, given $k < m'$ connected charities from A', there needs to be another one that can be reached in one step from one of the k charities.

This gives a system of linear equations where each $\delta^*(x_j)$ can be written in terms of $\delta^*(x_1)$: $\delta^*(x_2)$ only requires the two valuations v_{i_1,x_1} and v_{i_1,x_2}, $\delta^*(x_3)$ requires at most four valuations and so on. Considering this "worst case" in terms of representation length together with $\sum_{x \in A} \delta^*(x) = \sum_{i \in N'} C_i$, $\delta^*(x_1)$ can be written as the fraction of $\sum_{i \in N'} C_i$ and $1 + v_{i_1,x_2}/v_{i_1,x_1} + (v_{i_1,x_2}/v_{i_1,x_1}) \cdot (v_{i_2,x_3}/v_{i_2,x_2}) + \ldots$ resulting in a binary encoding length of $\log_2(C_N \cdot {v_{\max}}^{m-1})$ for the nominator and $\log_2(m \cdot {v_{\max}}^{m-1}))$ for the denominator where we upper-bounded m' by m, $\sum_{i \in N'} C_i$ by C_N, and $v_{i,x}$ by $v_{\max}$.

As x_1 was chosen arbitrarily and Leontief utilities coincide with distributions to certain charities, (δ^*, u^*) can be represented by $n + m$ times the derived length for $\delta^*(x_1)$. □

Lemma 18 cannot be used directly for computing the equilibrium distribution in polynomial time, since the proof requires us to know $T_{\delta^*,i}$. We cannot iterate over all possible $T_{\delta^*,i}$ as this would require exponential time.

To prove polynomial-time computability, we leverage Theorem 13 by Jain (2007):

Lemma 19 (Jain, 2007)**.** *Let S be a convex set given by a strong separation oracle, and $\phi > 0$ an integer.*

There is an oracle-polynomial time and ϕ-linear time algorithm which does one of the following:

1. *Concludes that there is no point in S with binary encoding length at most ϕ, or —*
2. *Produces a point in S with binary encoding length at most $P(n) \cdot \phi$, where $P(n)$ is a polynomial.*

We apply Lemma 19 as follows. For every positive rational number z_0, we define a convex set $S(z_0) \subseteq \mathbb{R}^{n+m}$, where the variables are u_i for $i \in N$ and d_x for $x \in A$:

$$\begin{aligned}
\prod_{i=1}^{n} {u_i}^{C_i} &\geq z_0; \\
d_x &\geq u_i \cdot v_{i,x} && \text{for all } i \in N, x \in A; \\
\sum_{x \in A} d_x &= C_N; \\
u_i &\geq 0 && \text{for all } i \in N; \\
d_x &\geq 0 && \text{for all } x \in A;
\end{aligned}$$

The set $S(z_0)$ represents all pairs (δ, u) such that δ is a feasible distribution, u is its utility profile, and the Nash product is at least z_0. A *strong separation oracle for* $S(z_0)$ is a function that accepts as input a rational vector $\mathbf{y}' = (u'_1, \ldots, u'_n, d'_1, \ldots, d'_m)$. It should return one of two outcomes: either an assertion that $\mathbf{y}' \in S(z_0)$, or a hyperplane that separates $\mathbf{y}'$ from $S(z_0)$ (that is, a rational vector c such that $c \cdot \mathbf{y}' < c \cdot \mathbf{y}$ for all $\mathbf{y} \in S(z_0)$).

Lemma 20. *For every rational $z_0 > 0$, there is a polynomial-time strong separation oracle for the convex set $S(z_0)$.*

Proof. Given a rational vector $\mathbf{y}' = (u'_1, \ldots, u'_n, d'_1, \ldots, d'_m)$, we first check whether the point satisfies the linear constraints $d'_x \geq u'_i \cdot v_{i,x}$, $\sum_{x \in A} d'_x = C_N$, $u'_i \geq 0$ and $d'_x \geq 0$. If one of these constraints is violated, the constraint itself yields a separating hyperplane. As the number of these constraints is polynomial in n, m, all of them can be checked in polynomial time.

It remains to handle the case that all linear constraints are satisfied, whereas the nonlinear constraint is violated. That is, we have

$$\prod_{i=1}^{n} (u'_i)^{C_i} < z_0. \tag{19}$$

Recall that we assume that all C_i are natural numbers; therefore the above condition can be checked exactly using arithmetic operations on rational numbers. The binary encoding length of the product is polynomial in the binary encoding length of u'_i, and in C_i; due to the latter fact, it is in fact pseudo-polynomial in the input size. However, if all contributions are equal, they can be ignored and the representation stays polynomial in the input size.

To construct a separating hyperplane for this case, we use an idea similar to the one in Jain (2007). We define the vector c to have the coefficient $\frac{1}{C_N} \cdot \frac{C_i}{u'_i}$ for each variable u_i, and the coefficient 0 for each variable d_x. Note that the encoding length of c is polynomial in the input size. For every vector $\mathbf{y} = (u_1, \ldots, u_n, d_1, \ldots, d_m)$,

$$c \cdot \mathbf{y} = \frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{u_i}{u'_i}.$$

Substituting $\mathbf{y} := \mathbf{y}'$ gives $c \cdot \mathbf{y}' = 1$. We now prove that $c \cdot \mathbf{y} > 1$ for every $\mathbf{y} \in S(z_0)$.

Indeed, $c \cdot \mathbf{y}$ is a weighted arithmetic mean of the n positive numbers $\frac{u_i}{u'_i}$, with weights C_i. By the weighted AM-GM inequality, this sum is at least as large as their weighted geometric mean, that is

$$\frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{u_i}{u'_i} \geq \left(\prod_{i=1}^{n} \left(\frac{u_i}{u'_i} \right)^{C_i} \right)^{1/C_N} = \frac{(\prod_{i=1}^{n} (u_i)^{C_i})^{1/C_N}}{(\prod_{i=1}^{n} (u'_i)^{C_i})^{1/C_N}}$$

Since $\mathbf{y} = (u_1, \ldots, u_n, d_1, \ldots, d_m)$ is in $S(z_0)$, it satisfies the inequality $(\prod_{i=1}^{n} (u_i)^{C_i}) \geq z_0$. Substituting in the right-hand side above gives:

$$\frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{u_i}{u'_i} \geq \frac{{z_0}^{1/C_N}}{(\prod_{i=1}^{n} (u'_i)^{C_i})^{1/C_N}} = \left(\frac{z_0}{\prod_{i=1}^{n} (u'_i)^{C_i}} \right)^{1/C_N},$$

which is larger than 1 by (19). Hence, $c \cdot \mathbf{y} > 1$, so c indeed defines a separating hyperplane. □

Lemma 20 allows us to apply Lemma 19 to $S(z_0)$. We can now compute the equilibrium distribution by applying binary search to z_0, in the following way.

1. Initialize $L :=$ the Nash product of some arbitrary distribution (e.g., the uniform distribution).

2. Initialize $H :=$ some upper bound on the optimal Nash product, e.g. the Nash product resulting from the (unrealistic) distribution in which each agent i divides C_N optimally (in proportion to v_i).

3. Let $\phi :=$ an upper bound on the binary encoding length of (δ^*, u^*), derived in Lemma 18.

4. Let $z_0 := (L + H)/2$. Note that both L and H can be encoded in length polynomial in the input size, so the same applies to z_0.

5. Apply Lemma 19 to $S(z_0)$, using Lemma 20 for the strong separation oracle.
   - If Lemma 19 yields outcome 1 ("no point in $S(z_0)$ with binary encoding length at most ϕ"), then we know that $S(z_0)$ does not contain an equilibrium distribution. This means that the Nash product of the equilibrium distribution is lower than z_0. We set $H := z_0$ and return to step 4.
   - If Lemma 19 yields outcome 2 ("a point in $S(z_0)$ with binary encoding length at most $P(n) \cdot \phi$"), then in particular we have a distribution δ with Nash product at least z_0.

     We check whether δ is an equilibrium distribution (this can be done in polynomial time). If it is, we return δ and finish. Otherwise, we set $L := z_0$ and return to step 4.

As the binary encoding length of (δ^*, u^*) is at most ϕ, the binary encoding length of the maximum Nash product is at most $\sum_i C_i \log_2(u_i^*) \leq C_N \cdot \phi$. Therefore, after at most $C_N \cdot \phi$ steps, the binary search is guaranteed to terminate with an equilibrium distribution.